Determinants

This is a number you get from a square matrix that has all sorts of magical properties. I'll define it by telling you how to compute it using row operations.

next 4 weeks

Def: The determinant of a square matrix A is a number $\det(A)$ or $|A|$ satisfying:

(1) $A \xrightarrow[\text{replacement}]{\text{row}} B$ then $\det(A) = \det(B)$

(2) $A \xrightarrow{R_i \times = c} B$ then $\det(A) = \frac{1}{c}\det(B)$

(3) $A \xrightarrow[\text{swap}]{\text{row}} B$ then $\det(A) = -\det(B)$

(4) $\det(I_n) = 1$.

Eg: $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \overset{R_3 \times = -1}{\underset{(2)}{=\!=}} -\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow \det(A) = 0$ if A has a zero row.

Eg: $\det\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \overset{R_1 \times = \frac{1}{a} \;\; R_2 \times = \frac{1}{b}}{\underset{R_3 \times = \frac{1}{c}}{=\!=}} abc \det\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

$\overset{\text{row}}{\underset{\text{replacements}}{=\!=}} abc \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{(4)}{=} abc$

$$\boxed{\det\begin{pmatrix} \text{triangular} \\ \text{matrix} \end{pmatrix} = \text{product of the diagonal entries}}$$

eg. REF

Eg: $\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix} \overset{R_2 -= R_1}{=\!=} \det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$\overset{R_3 -= R_2}{=\!=} \det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = 2$

Eg: $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \underset{R_3 -= 7R_1}{\overset{R_2 -= 4R_1}{=\!=}} \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$

$\overset{R_3 -= 2R_2}{=\!=} \det\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = 0$ zero row $\iff$ not invertible

fastest general algorithm

Procedure (Computing determinants by Elimination):

Run Gaussian elimination

(1) Row replacements don't change $\det(A)$

(2) $A \xrightarrow{R_i \times = c} B \implies \det(A) = \frac{1}{c}\det(B)$

(3) Row swaps negate the determinant

Once A is in REF, $\det(A)$ = product of diagonals.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$a \neq 0$: $\det(A) \overset{R_2 -= \frac{c}{a}R_1}{=\!=} \det\begin{pmatrix} a & b \\ 0 & d - \frac{c}{a} \end{pmatrix} = a(d - \frac{c}{a})$

$= ad - bc$

$a = 0$: $\det(A) \overset{R_1 \leftrightarrow R_2}{=\!=} -\det\begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc = ad - bc$

$$\boxed{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc}$$

Why do you get the same number with different row operations?

Eg: $\det\begin{pmatrix}1&1&0\\1&2&1\\0&1&3\end{pmatrix} \xlongequal{R_2 - = R_1} \det\begin{pmatrix}1&1&0\\0&1&1\\0&1&3\end{pmatrix}$

$\xlongequal{R_2 \leftrightarrow R_3} -\det\begin{pmatrix}1&1&0\\0&1&3\\0&1&1\end{pmatrix} \xlongequal{R_3 - = R_2} -\det\begin{pmatrix}1&1&0\\0&1&3\\0&0&-2\end{pmatrix}$

$= -(-2) = 2$ ✓

Magical Properties of the Determinant

Existence: There is a number $\det(A)$ satisfying (1)–(4).

Invertibility: $\det(A) \neq 0 \iff A$ is invertible, in which case, $\det(A^{-1}) = 1/\det(A)$

Multiplicativity: $\det(AB) = \det(A)\det(B)$.

Transposes: $\det(A^T) = \det(A)$

Multilinearity:

$$\det\begin{pmatrix}| & | & | \\ v_1 & av_2 + bv_2' & v_3 \\ | & | & |\end{pmatrix} = a\det\begin{pmatrix}| & | & | \\ v_1 & v_2 & v_3 \\ | & | & |\end{pmatrix} + b\det\begin{pmatrix}| & | & | \\ v_1 & v_2' & v_3 \\ | & | & |\end{pmatrix}$$

& likewise for rows.

Eg: $\det\begin{pmatrix}1&3&4\\2&0&2\\1&2&3\end{pmatrix} = 0$ b/c (col 1) + (col 2) = (col 3)

Eg: $\det\left[\begin{pmatrix}1&1&0\\1&2&1\\0&1&3\end{pmatrix}^{100}\right] = \left[\det\begin{pmatrix}1&1&0\\1&2&1\\0&1&3\end{pmatrix}\right]^{100} = 2^{100}$

$\det(A^{100}) = \det(A \cdot A^{99}) = \det(A)\det(A^{99}) = \cdots = \det(A)^{100}$

Transpose property: det satisfies (1)–(3) for column ops (row ops on A^T)

Eg: $\det\begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \overset{C_1 -= 4C_3}{=\!=} \det\begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

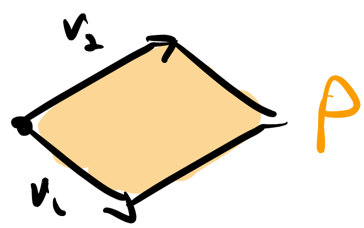
$\overset{C_1 += 9C_2}{=\!=} \det\begin{pmatrix} 49 & 7 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = 49$

Determinants & Volumes

Where do properties (1)–(4) come from?

Two vectors $v_1, v_2 \in \mathbb{R}^2$ determine ("span") a parallelogram.

$P = \{x_1 v_1 + x_2 v_2 : x_1, x_2 \in [0,1]\}$

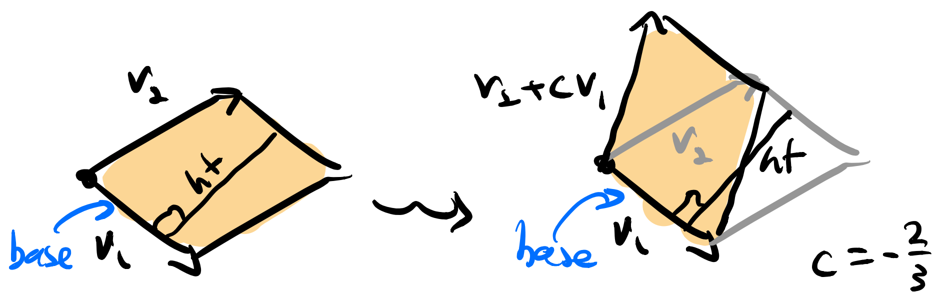


Fact: $\text{area}(P) = \left|\det\begin{pmatrix} -v_1^T- \\ -v_2^T- \end{pmatrix}\right| = |ad - bc|$

Why?

(1) Row replacement

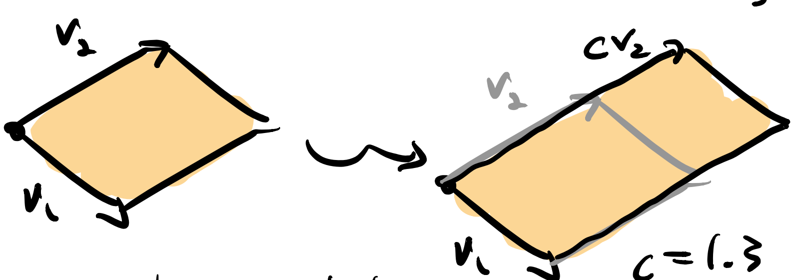
$v_2 \rightsquigarrow v_2 + cv_1$



area = base × ht : unchanged

(2) Row scaling

$v_2 \rightsquigarrow cv_2$

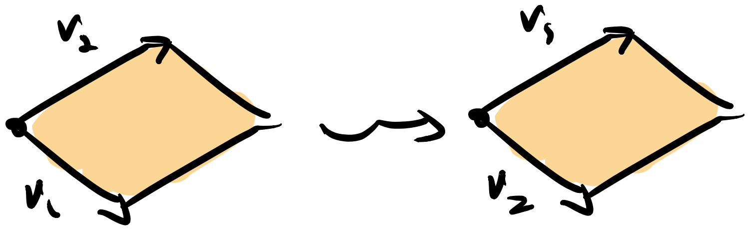


ht scaled by $|c| \Rightarrow$ base × ht : scaled by $|c|$

(3) Row Swap

$v_1 \longleftrightarrow v_2$

area unchanged $= |\det|$

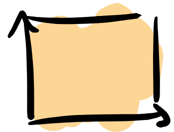


Q: minus sign?

HW #8.17

(4) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

area $= 1$.

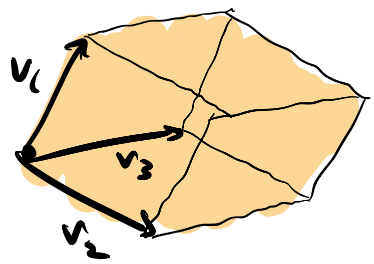


This generalizes as follows (same reasoning):

Def: The parallelepiped determined ("spanned") by n vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ is

$$P = \{x_1 v_1 + \cdots + x_n v_n : x_1, \ldots, x_n \in [0,1]\}$$

P



Thm (Determinants & Volumes):

$$\text{volume}(P) = \left| \det \begin{pmatrix} — v_1^T — \\ \vdots \\ — v_n^T — \end{pmatrix} \right|$$

Q: When is volume$(P) = 0$? When $\{v_1, \ldots, v_n\}$ is LD

($\Longleftrightarrow \det = 0$)

Notes:

(1) "volume" in $\mathbb{R}^2$ = area

(2) Used in change of variables formula in multivariable calculus $du_1 \cdots du_n = \det\left(\partial u_i / \partial x_j\right) dx_1 \cdots dx_n$