Cofactor Expansion

This is a handy recursive formula for $\det(A)$.

Def: Let A be an $n \times n$ matrix.

- The (i,j) minor A_{ij} is obtained by deleting the i^{th} row & j^{th} column of A. $(n-1)\times(n-1)$ matrix
- The (i,j) cofactor C_{ij} is
$$C_{ij} = (-1)^{i+j}\det(A_{ij}).$$
- The cofactor matrix is the matrix C whose (i,j) entry is C_{ij}.

Eg: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ $\quad A_{21} = \begin{pmatrix} \cancel{1} & 1 & 0 \\ \cancel{1} & \cancel{2} & \cancel{1} \\ \cancel{0} & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix}$

$$C_{21} = (-1)^{2+1}\det\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} = -3$$

NB: $(-1)^{i+j}$ follows this pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Thm (Cofactor Expansion):

Let $A = (a_{ij})$ be an $n \times n$ matrix, C_{ij} = cofactors

(1) Cofactor expansion along the i^{th} row:
$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(1) Cofactor expansion along the j^{th} column:
$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Eg: $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Expand along the 3rd row:

$$\det(A) = 0 \cdot \det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + 3 \det\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= 0 - 1(1) + 3(2-1) = -1 + 3 = 2$$

Expand along 1st column:

$$\det(A) = 1 \cdot \det\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} - 1 \det\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} + 0 \det\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$= 1(6-1) - 1(3) + 0 = 5 - 3 = 2$$

Remarks:

(1) This is a recursive formula: $C_{ij} = \det((n-1)\times(n-1))$

(2) You can calculate $C_{ij} = (-1)^{i+j}\det(A_{ij})$ using any method you like — you'll always get the same number.

(3) Expanding along any row/column gives you the determinant — same number!

(4) This is handy when A has unknown entries or if there's a row or column with lots of zeros. (Otherwise ridiculously inefficient: $O(n! \cdot n)$)

Eg:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - ge)$$

$= aei + bfg + cdh - afh - bdi - ceg$

How to remember this?

Sarrus' Scheme:

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

To compute det(3x3 matrix):

$\det = aei + bfg + cdh$
$\quad - ceg - afh - bdi$

Eg:

$$\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 1 & 3 & 0 & 1 \end{array}$$

$\det = 1\cdot 2\cdot 3 + 1\cdot 1\cdot 0 + 0\cdot 1\cdot 1$
$\quad - 0\cdot 2\cdot 0 - 1\cdot 1\cdot 1 - 3\cdot 1\cdot 1$
$= 6 - 1 - 3 = 2$ ✓

Warning: This only works for 3x3 matrices!

Eg:

$$\det\begin{pmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 0 \\ -1 & 6 & 4 & 0 \end{pmatrix}$$

column with lots of zeros

$$= -(-2)\det\begin{pmatrix} -2 & -3 & 2 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix} - 5\det\begin{pmatrix} 2 & 5 & -3 \\ 1 & 3 & -2 \\ -1 & 6 & 4 \end{pmatrix}$$

$$-0\cdot\det\begin{pmatrix}\text{don't} \\ \text{care}\end{pmatrix} + 0\det\begin{pmatrix}\text{don't} \\ \text{care}\end{pmatrix}$$

$$= 2(-24) - 5(11) = -48 - 55 = -103$$

only computed two 3x3 dets

Methods for Computing Determinants

(1) Special formulas $(2\times 2,\ 3\times 3)$

$\to$ best for small matrices, except 3×3 with lots of 0's

(2) Cofactor expansion

$\to$ best if you have unknown entries, or a row/column with lots of zeros.

(3) Row (& column) operations

$\to$ best if you have a big matrix with no unknown entries & no row or column with lots of zeros.

(4) Any combination of the above

$\to$ eg. do a row op. to create a column with lots of zeros, then expand cofactors, ...

Thm: Let C be the cofactor matrix of A. Then

$$AC^T = \det(A) I_n = C^T A$$

In particular, if $\det(A) \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{\det(A)} C^T}$$

see supplement

$\rightsquigarrow$ Ridiculously inefficient computationally.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\rightsquigarrow$ generalizes the formula for 2×2 inverse

Cross Products

This is a trick for vectors in $\mathbb{R}^3$. $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Def: If $v = (a\ b\ c)$, $w = (d\ e\ f)$ then

$$v \times w = \text{"}\det\begin{pmatrix} e_1 & e_2 & e_3 \\ a & b & c \\ d & e & f \end{pmatrix}\text{"} = (bf - ec)e_1 - (af - cd)e_2 + (ae - bd)e_3$$

is their cross product.

$$= \begin{pmatrix} bf - ec \\ cd - af \\ ae - bd \end{pmatrix} \in \mathbb{R}^3$$

Eg:
$$\begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\0\\1\end{pmatrix} = \text{"}\det\begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\text{"} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

Def: For $u, v, w \in \mathbb{R}^3$ their triple product is

$$u \cdot (v \times w) = \det\begin{pmatrix} - & u^T & - \\ - & v^T & - \\ - & w^T & - \end{pmatrix}$$

$\rightarrow$ another formula for $\det(3\times 3)$.

Eg: $u = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $v = \begin{pmatrix}1\\2\\1\end{pmatrix}$ $w = \begin{pmatrix}0\\1\\3\end{pmatrix}$

$$v \times w = \text{"}\det\begin{pmatrix} e_1 & e_2 & e_3 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}\text{"} = \begin{pmatrix}5\\-3\\1\end{pmatrix}$$

$$u \cdot (v \times w) = \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}5\\-3\\1\end{pmatrix} = 5 - 3 = 2 = \det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$ ✓

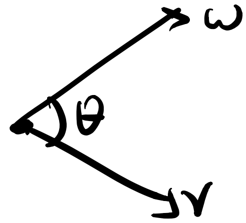
Properties:

(1) $v \times w \perp v$, $v \times w \perp w$

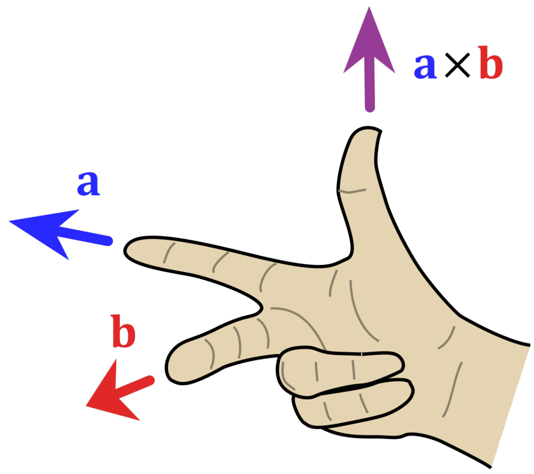
$$\rightarrow v \cdot (v \times w) = \det \begin{pmatrix} - & v^T & - \\ - & v^T & - \\ - & w^T & - \end{pmatrix} = 0$$

(2) $\|v \times w\| = \|v\| \cdot \|w\| \cdot \sin(\theta)$

$\rightarrow$ compare $\|v \cdot w\| = \|v\| \cdot \|w\| \cos(\theta)$



(3) $v \times w$ points in the direction determined by the right hand rule.



(1)-(3) characterize $v \times w$

(4) $v \times w = 0 \iff v, w$ are collinear

$\rightarrow \theta = 0°, 180° \implies \sin(\theta) = 0$

(5) $w \times v = -v \times w$

$$\rightarrow \det \begin{pmatrix} e_1 & e_2 & e_3 \\ - & w^T & - \\ - & v^T & - \end{pmatrix} = -\det \begin{pmatrix} e_1 & e_2 & e_3 \\ - & v^T & - \\ - & w^T & - \end{pmatrix}$$