Eigenvalues & Eigenvectors

Core concept in linear algebra.

- → Difference equations
- → Differential equations
- → Markov chains
- → SVD (later)

Motivating Problem

In a population of rabbits:

- 1/4 survive their 1st year
- 1/2 survive their 2nd year
- Max lifespan is 3 years
- 1, 2-yr-old rabbits have 13, 12 babies

translates to

Q: What's the long-term behavior of the system?

x_{2020} = # babies in 2020 $\qquad x_{2021} = 13y_{2020} + 12z_{2020}$

y_{2020} = # 1st year in 2020 $\qquad y_{2021} = \frac{1}{4}x_{2020}$

z_{2020} = # 2nd year in 2020 $\qquad z_{2021} = \frac{1}{2}y_{2020}$

Encode the three quantities (state of the system) in a vector:

$$v_{2020} = \begin{pmatrix} x_{2020} \\ y_{2020} \\ z_{2020} \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

state change matrix

Then $v_{2021} = Av_{2020}$, $v_{2022} = Av_{2021} = A^2 v_{2020}$,

$v_{2023} = A v_{2022} = A^2 v_{2021} = A^3 v_{2020}$, etc.

This is called a difference equation: the next state is (matrix)x(previous state):

$$\boxed{v_{n+1} = A v_n}$$

Q: What's the long-term behavior of the system?

$\to v_n = A^n v_0$, but don't want to compute A^{1000} — gives no qualitative information.

Observation:

$$w_1 = \begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} \quad Aw_1 = \begin{pmatrix} 64 \\ 8 \\ 2 \end{pmatrix} = 2w_1$$

$$w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad Aw_2 = \begin{pmatrix} -1 \\ 1/2 \\ -1/2 \end{pmatrix} = -\tfrac{1}{2}w_2$$

$$w_3 = \begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix} \quad Aw_3 = \begin{pmatrix} -27 \\ 9/2 \\ -3/2 \end{pmatrix} = -\tfrac{3}{2}w_3$$

$A\cdot(-)$ scales these vectors!!!

Say we started with $v_0 = (16, 6, 1)$ rabbits.

Solve $v_0 = x_1 w_1 + x_2 w_2 + x_3 w_3 \rightsquigarrow x_1 = 1 \quad x_2 = 1 \quad x_3 = -1$

$$v_0 = w_1 + w_2 - w_3$$

$$v_1 = Av_0 = A(w_1 + w_2 - w_3) = Aw_1 + Aw_2 - Aw_3 = 2w_1 - \tfrac{1}{2}w_2 + \tfrac{3}{2}w_3$$

$$v_2 = Av_1 = A(2w_1 - \tfrac{1}{2}w_2 + \tfrac{3}{2}w_3) = 2Aw_1 - \tfrac{1}{2}Aw_2 + \tfrac{3}{2}Aw_3 = 2^2 w_1 + (-\tfrac{1}{2})^2 w_2 - (-\tfrac{3}{2})^2 w_3$$

$\vdots$

dominates

$$v_n = A^n v_0 = A^n(w_1 + w_2 - w_3) = A^n w_1 + A^n w_2 - A^n w_3 = 2^n w_1 + (-\tfrac{1}{2})^n w_2 - (-\tfrac{3}{2})^n w_3$$

The $2^n w_1$ term dominates:

$$v_n \approx 2^n w_1 :$$

eventually v_n = scalar multiple of $w_1 = (32, 4, 1)$: ratios $32:4:1$. eventually v_n doubles each year.

Key ingredient: A acts on w_1, w_2, w_3 by scaling.

Def: An eigenvector of a square matrix A is a nonzero vector v s.t. Av is a scalar multiple of v:

$$Av = \lambda v \qquad \lambda = \text{eigenvalue}$$

Eg:
$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\underbrace{\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}}_{\text{eigenvector}} = \begin{pmatrix} 64 \\ 8 \\ 2 \end{pmatrix} = \underbrace{2}_{\text{eigenvalue}}\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}$$

Geometrically, an eigenvector is a vector v st. Av lies on the line thru v.

How to compute?

v is an eigenvector of A w/ eigenvalue λ

$$\iff Av = \lambda v \qquad (v \neq 0)$$
$$\iff Av - \lambda I_n v = 0 \qquad (v \neq 0)$$
$$\iff (A - \lambda I_n)v = 0 \qquad (v \neq 0)$$
$$\iff v \in \mathrm{Nul}(A - \lambda I_n) \qquad (v \neq 0)$$

Def: Let λ be an eigenvalue of A. The λ-eigenspace is $\mathrm{Nul}(A - \lambda I_n) = \{v \in \mathbb{R}^n : Av = \lambda v\} = \left\{\begin{matrix}\text{all } \lambda\text{-eigenvectors} \\ \text{and } 0\end{matrix}\right\}$

Eg: $A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ $\lambda = 2$ $A - 2I_3 = \begin{pmatrix} -2 & 13 & 12 \\ 1/4 & -2 & 0 \\ 0 & 1/2 & -2 \end{pmatrix}$

$$\begin{pmatrix} -2 & 13 & 12 \\ 1/4 & -2 & 0 \\ 0 & 1/2 & -2 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & -32 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{PVF}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}$$

So 2-eigenspace is $\mathrm{Span}\left\{\begin{pmatrix}32\\4\\1\end{pmatrix}\right\}$

NB: If λ is an eigenvalue $\Longrightarrow$ eigenvectors (and 0) form a nonzero subspace. (∞ eigenvectors)

NB: If $A-\lambda I$ is invertible (or FCR, or $\mathrm{Nul} = \{0\}$)
$\Longrightarrow$ no λ-eigenvectors $\Rightarrow \lambda$ is not an eigenvalue.

Eg: $A = \begin{pmatrix}-1 & 0 & 0\\ -1 & 0 & 2\\ -1 & 1 & 1\end{pmatrix}$ $\lambda = -1$ $A - \lambda I = \begin{pmatrix}0 & 0 & 0\\ -1 & 1 & 2\\ -1 & 1 & 2\end{pmatrix}$

$\xrightarrow{\text{RREF}} \begin{pmatrix}1 & -1 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \xrightarrow{\text{PVF}} \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_2\begin{pmatrix}1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}2\\0\\1\end{pmatrix}$

So (-1)-eigenspace is $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right\}$

Eg: $A = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$ $\lambda = 0$ $A - \lambda I = A$

0-eigenspace $= \mathrm{Nul}(A - 0I) = \mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}\right\}$

0 is an eigenvalue
$\Longleftrightarrow \mathrm{Nul}(A) \neq \{0\} \Longleftrightarrow A$ is not invertible
in which case the 0-eigenspace is $\mathrm{Nul}(A)$

Eg: Let V be a subspace of $\mathbb{R}^n$, P_V = projection matrix.

Q: What are all eigenvectors?

0-eigenspace: $\mathrm{Nul}(P_V) = V^\perp$

1-eigenspace: $P_V x = 1x \iff x \in V$, so it's V.

Eg: $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\theta = 30°$ rotation

Q: What are all eigenvectors?

None! (no real eigenvectors)

Rotate by 30° $\rightsquigarrow$ not on same line.

Eg: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ shear $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

Q: What are all eigenvectors?

Only the x-axis (eigenvalue = 1)