Last time: wanted to solve $Av = \lambda v$

eigenvector (v), eigenvalue (λ)

If λ is an eigenvalue then the

λ-eigenspace $= \{$all λ-eigenvectors $\& \, 0\} = \mathrm{Nul}(A - \lambda I_n)$

How to compute eigenvalues?

Characteristic Polynomial

λ is an eigenvalue

$\iff$ $Av = \lambda v$ has a nonzero solution

$\iff$ $(A - \lambda I_n)v = 0$ has a nonzero solution

$\iff$ $A - \lambda I_n$ is not invertible

$\iff$ $p(\lambda) = \det(A - \lambda I_n) = 0$

Eg: $A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ $\quad A - \lambda I_3 = \begin{pmatrix} -\lambda & 13 & 12 \\ 1/4 & -\lambda & 0 \\ 0 & 1/2 & -\lambda \end{pmatrix}$

$$p(\lambda) = \det(A - \lambda I_3) = -\lambda \det\begin{pmatrix} -\lambda & 0 \\ 1/2 & -\lambda \end{pmatrix} - \frac{1}{4}\det\begin{pmatrix} 13 & 12 \\ 1/2 & -\lambda \end{pmatrix}$$

$$= -\lambda^3 - \frac{1}{4}(-13\lambda - 6) = -\lambda^3 + 13/4\,\lambda + 3/2$$

What are the zeros of $p(\lambda)$?

Aside:

- Real life: ask a computer
- The computer will turn this back into an eigenvalue problem & use a different (faster) eigenvalue-finding algorithm.

- By hand: I'll give you one root λ_0. Compute $p(\lambda)/(\lambda-\lambda_0)$ = deg 2 poly.

Eg cont'd: fact: 2 is a root of $-\lambda^3+{}^{13}/_4\lambda+{}^3/_2$.

Synthetic division:

$$
\begin{array}{r}
-\lambda^2 - 2\lambda - 3/4 \\
\lambda-2 \,\overline{)\, -\lambda^3 + 13/4\,\lambda + 3/2} \\
-(-\lambda^3+2\lambda^2) \\
\hline
-2\lambda^2 + 13/4\,\lambda + 3/2 \\
-(-2\lambda^2+4\lambda) \\
\hline
-3/4\,\lambda + 3/2 \\
-(-3/4\,\lambda + 3/2) \\
\hline
0
\end{array}
$$

$\Rightarrow p(\lambda) =$

$$(\lambda-2)\underbrace{(-\lambda^2-2\lambda-3/4)}_{\text{Quadratic formula}}$$

$$\lambda = -\tfrac{1}{2}\left(2\pm\sqrt{4-3}\right)$$
$$= -\tfrac{1}{2}(2\pm 1) = -\tfrac{1}{2}, -\tfrac{3}{2}$$

$\Rightarrow p(\lambda) = -(\lambda-2)(\lambda+\tfrac{1}{2})(\lambda+3/2)$

$\Rightarrow$ eigenvalues are $2, -1/2, -3/2$

Compute eigenvectors by finding bases for $\mathrm{Nul}(A-\lambda I_3)$

$2: \; w_1 = \begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} \qquad -\tfrac{1}{2}: \; w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad -\tfrac{3}{2}: \; w_3 = \begin{pmatrix} 18 \\ -3 \\ 1 \end{pmatrix}$

Explains where these came from last time!

Def: The characteristic polynomial of an $n\times n$ matrix A is
$$p(\lambda) = \det(A - I_n\lambda)$$

$$\boxed{\lambda \text{ is an eigenvalue of } A \iff p(\lambda)=0}$$

What does $p(\lambda)$ look like?

Def: The **trace** of a matrix is
$Tr(A)$ = sum of diagonal entries.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad A - \lambda I_2 = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

$$p(\lambda) = \det(A - \lambda I_2) = (a-\lambda)(d-\lambda) - bc = \lambda^2 - \underbrace{(a+d)}_{Tr}\lambda + \underbrace{(ad-bc)}_{\det}$$

$$\boxed{p(\lambda) = \lambda^2 - Tr(A)\lambda + \det(A) \quad \text{for } 2\times 2 \ A}$$

General form: If A is an $n\times n$ matrix,

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} Tr(A) \lambda^{n-1} + \cdots + \det(A)$$

more complicated terms

→ constant term is $p(0) = \det(A - 0 I_n) = \det(A)$ ✓
→ $p(\lambda)$ is a polynomial of degree n

Consequence: An $n\times n$ matrix has at most n eigenvalues.

Diagonalization

In our rabbit problem, turns out: $\{w_1, w_2, w_3\}$ is LI
$\Rightarrow$ form a **basis** for $\mathbb{R}^3$.

We wanted to understand $v_n = A^n v_0$.

$v_0 = x_1 w_1 + x_2 w_2 + x_3 w_3$ has a (unique) solution!

This was an important step in solving the problem.

Def: Let A be an $n\times n$ matrix.

(1) An **eigenbasis** is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

(2) A is **diagonalizable** if there is an eigenbasis.

If A is diagonalizable $\rightsquigarrow$ eigenbasis $\{w_1, \dots, w_n\}$ w/ eigenvalues $\lambda_1, \dots, \lambda_n$. To compute $A^n v$:

(1) Solve $v = x_1 w_1 + \cdots + x_n w_n$

(2) $\boxed{A^m v = \lambda_1^m x_1 w_1 + \cdots + \lambda_n^m x_n w_n}$ ← vector form

Eg: In the rabbit problem,

$$v_0 = (16, 6, 1) \overset{(1)}{=} w_1 + w_2 - w_3 \qquad (x_1 = 1 \quad x_2 = 1 \quad x_3 = -1)$$

$$v_n = A^n v_0 = A^n(w_1 + w_2 - w_3) = 2^n w_1 + \left(-\tfrac{1}{2}\right)^n w_2 + \left(-\tfrac{3}{2}\right)^n w_3$$

Matrix Form of Diagonalization

Let $\{w_1, \dots, w_n\}$ be an eigenbasis for A, $\lambda_1, \dots, \lambda_n$ = eigenvals

$$\Rightarrow \boxed{A = CDC^{-1} \qquad C = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_n \\ | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}$$

Conversely, if $A = CDC^{-1}$ for D diagonal, then the columns of C form an eigenbasis, & the diagonal entries of D are the corresponding eigenvalues.

Why? $C\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = x_1w_1+\cdots+x_nw_n \Rightarrow C^{-1}(x_1w_1+\cdots+x_nw_n)=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$

$$CDC^{-1}(x_1w_1+\cdots+x_nw_n) = CD\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = C\begin{pmatrix}\lambda_1x_1\\ \vdots\\ \lambda_nx_n\end{pmatrix}$$

$$= \lambda_1x_1w_1+\cdots+\lambda_nx_nw_n = A(x_1w_1+\cdots+x_nw_n)$$

Eg: $\begin{pmatrix}0 & 13 & 12\\ 1/4 & 0 & 0\\ 0 & 1/2 & 0\end{pmatrix} = CDC^{-1}$ $\quad C = \begin{array}{c} \begin{matrix} w_1 & w_2 & w_3\end{matrix} \\ \begin{pmatrix}32 & 2 & 18\\ 4 & -1 & -3\\ 1 & 1 & 1\end{pmatrix}\end{array}$ $\quad D = \begin{pmatrix}2 & 0 & 0\\ 0 & -\frac{1}{2} & 0\\ 0 & 0 & -\frac{3}{2}\end{pmatrix}$

NB: $A^m = (CDC^{-1})^m = CD\cancel{C^{-1}C}D\cancel{C^{-1}C}D\cancel{C^{-1}}\cdots\cancel{C}DC^{-1} = CD^mC^{-1}$

$= C\begin{pmatrix}\lambda_1^m & & 0\\ & \ddots & \\ 0 & & \lambda_n^m\end{pmatrix}C^{-1}$: much easier to compute powers of A.

Eg: $A=\begin{pmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 4\end{pmatrix}$ diagonal matrix: eigenvectors are coordinate vectors

$Ae_1 = 2e_1 \quad Ae_2 = 3e_2 \quad Ae_3 = 4e_3$

$\Rightarrow A = CDC^{-1} \quad C = I_3 \quad D = A$

Eg: $A = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ shear

$p(\lambda) = \lambda^2 - Tr(A) + \det(A) = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2$

One eigenvalue $\lambda = 1$. 1-eigenspace: $\mathrm{Nul}(A - I_2)$

$A - I_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \xrightarrow{\text{PVF}} \begin{pmatrix}x\\ y\end{pmatrix} = x\begin{pmatrix}1\\ 0\end{pmatrix}$

1-eigenspace is the x-axis.

$\Rightarrow$ no eigenbasis! $\Rightarrow$ not diagonalizable.

Procedure to Diagonalize a Matrix:

(1) Compute the characteristic polynomial $p(\lambda)$
(2) Find the roots of $p(\lambda)$ = eigenvalues of A
(3) Find a basis for each eigenspace $= \mathrm{Nul}(A - \lambda I_n)$
(using PVF)

If you end up with n vectors in all your bases, they form an eigenbasis. Otherwise, not diagonalizable.

Fact: If $w_1, \dots, w_n$ are eigenvectors with different eigenvalues then $\{w_1, \dots, w_n\}$ is linearly independent.

So in the procedure you never have to check LI of bases of different eigenspaces.

Consequence: If A has n different eigenvalues, then A is diagonalizable (each eigenval has at least 1 eigenvec).