# WEEK 5 SUPPLEMENT

## 1. Full Rank Matrices

With the notions of dimension and linear independence in place, we now have a large number of ways to say that a matrix has full column rank / full row rank. We have covered in class the reasons why the Theorems below are true; I highly recommend making sure you're comfortable with those reasons.

We start with full column rank:

**Theorem.** *Let $A$ be an $m \times n$ matrix. The following are equivalent:*[1]

1. *$A$ has full column rank.*
2. *$A$ has a pivot in every column.*
3. *$A$ has no free columns.*
4. *$A$ has rank $n$.*
5. $\dim \operatorname{Col}(A) = n$.
6. *$Ax = 0$ has only the trivial solution.*
7. *$Ax = b$ has $0$ or $1$ solution for every $b$.*
8. $\operatorname{Nul}(A) = \{0\}$.
9. *The columns of $A$ are linearly independent.*

Full row rank is a complementary concept:

**Theorem.** *Let $A$ be an $m \times n$ matrix. The following are equivalent:*

1. *$A$ has full row rank.*
2. *$A$ has a pivot in every row.*
3. *A row echelon form of $A$ has no zero rows.*
4. *$A$ has rank $m$.*
5. $\dim \operatorname{Col}(A) = m$.
6. $\operatorname{Col}(A) = \mathbf{R}^m$.
7. *$Ax = b$ is consistent for every $b$.*
8. $\dim \operatorname{Nul}(A) = n - m$.
9. *The columns of $A$ span $\mathbf{R}^m$.*

[1]This means that, for a given matrix $A$, either *all* of the statements are true, or *all* of them are false.

A *square* matrix has full column rank if and only if it has full row rank, if and only if it is invertible. There are some other equivalent conditions for a square matrix to be invertible:

**Theorem.** *Let $A$ be a* square *matrix of size $n \times n$. The following are equivalent:*

(1) *$A$ is invertible.*
(2) *$A$ has full column rank.*
(3) *$A$ has full row rank.*
(4) *The reduced row echelon form of $A$ is $I_n$.*
(5) *There is an $n \times n$ matrix $B$ such that $AB = I_n$ (a "right inverse").*
(6) *There is an $n \times n$ matrix $B$ such that $BA = I_n$ (a "left inverse").*
(7) *$A^T$ is invertible.*

## 2. The Basis Theorem

The following theorem is useful for verifying that a set of vectors forms a basis of a subspace.

**Theorem** (The Basis Theorem)**.**
*Let $V$ be a subspace of dimension $d$, and let $v_1, \ldots, v_d$ be vectors in $V$.*

(1) *If $\{v_1, \ldots, v_d\}$ is linearly independent, then it is a basis for $V$.*
(2) *If $\{v_1, \ldots, v_d\}$ is spans $V$, then it is a basis for $V$.*

In other words, if you know the dimension of a subspace, and you have the correct number of vectors in the subspace, then in order to check that subset is a basis, you have to verify that it spans *or* that it is linearly independent (not both). For instance, any two linearly independent vectors span a plane, and any two vectors spanning a plane are linearly independent.