Math 218D Problem Session

Week 11

1. Shape of quadratic forms

For each of the following quadratic forms:

(1) Plot the equation $q(x, y) = 1$ using a computer, and describe the shape (for example, for **a)** you should get an ellipse in $\mathbf{R}^2$, not an elliptic paraboloid in $\mathbf{R}^3$).

(2) Find the 2×2 symmetric matrix $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that

$$q(x, y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

(3) Recall that a symmetric matrix is **positive-definite** if all of its eigenvalues are positive. Test if the symmetric matrix S is positive-definite or not using the **pivot test**: Put S into REF without doing row-swaps or scaling. (If you need to do a row-swap, the matrix is not positive-definite.) If the diagonal entries of the REF are all positive, then S is positive-definite.

(4) What does the positive-definiteness of S have to do with the shape from (1)? You may need to do many examples until you see the pattern.

a) $q(x, y) = 2x^2 + 3y^2$ has $S = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, which is positive-definite, and $q = 1$ is an ellipse.

b) $q(x, y) = x^2 - 5y^2$ has $S = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

c) $q(x, y) = y^2$ has $S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is two lines.

d) $q(x, y) = -3x^2 - 2y^2$ has $S = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix}$, is not positive-definite, and $q = 1$ is empty.

e) $q(x, y) = x^2 + 3xy + y^2$ has $S = \begin{pmatrix} 1 & 3/2 \\ 3/2 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

f) $q(x, y) = 2x^2 + 4xy + y^2$ has $S = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is a hyperbola.

g) $q(x, y) = x^2 - 4xy + 5y^2$ has $S = \begin{pmatrix} 1 & -2 \\ -2 & \\ 5 & \end{pmatrix}$, is positive-definite, and $q = 1$ is an ellipse.

h) $q(x,y,z) = x^2+y^2+z^2+xy+yz+xz$ has $S = \begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$, is positive-definite, and $q = 1$ is an ellipsoid.

i) $q(x,y,z) = x^2+y^2+z^2+2xy+2yz+2xz$ has $S = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, is not positive-definite, and $q = 1$ is two planes.

2. Diagonalizing quadratic forms

Consider the quadratic form

$$q(x,y) = \frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2.$$

a) $S = \begin{pmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{pmatrix}$

b) $S = (\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix})\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}(\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix})^{-1}$

c) The ellipse $q(x,y) = 1$ is a rotated version of the ellipse $4x_0^2 + 1y_0^2 = 1$.

d) $(x_0, y_0) = Q^T(x,y) = ((1/\sqrt{2})x + (1/\sqrt{2})y, (-1/\sqrt{2})x + (1/\sqrt{2})y)$.

e) In terms of equations and not pictures, we can see that $4x_0^2 + 1y_0^2 = 1$ is an ellipse since both 4 and 1 are positive. Since the change of variables $(x_0, y_0) = ((1/\sqrt{2})x + (1/\sqrt{2})y, (-1/\sqrt{2})x + (1/\sqrt{2})y))$ corresponds to a rotation (Q is a rotation matrix!), this means that $q(x,y) = 1$ is a rotated ellipse.

f) The function $q(x,y) = 4((1/\sqrt{2})x + (1/\sqrt{2})y)^2 + ((-1/\sqrt{2})x + (1/\sqrt{2})y)^2$ is non-negative, as it is a sum of squares with positive coefficients. If it were equal to zero, then both $(1/\sqrt{2})x + (1/\sqrt{2})y$ and $(1/\sqrt{2})x + (1/\sqrt{2})y$ would need to equal zero - but this would mean that $x = y = 0$.

g) The major axis has length $1/\sqrt{\lambda_2} = 1$ and the minor axis has length $1/\sqrt{\lambda_1} = 1/2$. One explanation for this is that you can check the length of the axis of the ellipse $4x_0^2 + 1y_0^2 = 1$ by finding the x_0 and y_0 intercepts (as an ellipse in the (x_0, y_0) plane).

h) The direction of the major axis is the second eigenvector $1/\sqrt{2}(-1,1)$, while the direction of the minor axis is the first eigenvector $1/\sqrt{2}(1,1)$.

i) The maximum value of $q(x,y) = 1$, constrained to $||(x,y)|| = 1$, is the larger eigenvalue, 4, and is achieved at $(x,y) = \pm 1/\sqrt{2}(1,1)$. The minimum value of $q(x,y) = 1$, constrained to $||(x,y)|| = 1$, is the smaller eigenvalue, 1, and is achieved at $(x,y) = \pm 1/\sqrt{2}(-1,1)$.

3. LDL^T **decomposition**

a) $S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. This has REF (no scaling or swapping) given by $U = \begin{pmatrix} 2 & 1 \\ 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3/2 \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} = DL^T$. Therefore $S = \begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/2 & 1 \end{pmatrix}^T$.

b) $S = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$ has REF $U = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}^T$.

4. Relation to the quadratic formula

For 2×2 symmetric matrices $S = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$, there is an easy test for positive-definiteness, the **discriminant test**:

S is positive-definite if and only if both $a > 0$ and $b^2 - 4ac < 0$.

Let's verify this test in two ways, by relating it to other tests.

a) Method one: Relate the discriminant test to the **determinant test**: S is positive-definite if and only if $\det((a)) > 0$ and $\det(\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}) > 0$.

The first determinant condition is just $a > 0$. The second determinant is $ac - (1/4)b$. This is positive if and only if $b^2 - 4ac < 0$.

b) Method two:

(1) The quadratic form $q(x,y) = (x,y)^T S(x,y)$ equals

$$q(x,y) = ax^2 + bxy + cy^2$$

and factors into

$$q(x,y) = a(x - \frac{-b + \sqrt{b^2 - 4ac}}{2}y)(x - \frac{-b - \sqrt{b^2 - 4ac}}{2}y).$$

You can verify this factorization using the quadratic formula (pretend y is a number, and find the two roots of $ax^2 + (by)x + (cy^2)$: $x = \frac{-by \pm \sqrt{b^2y^2 - 4acy^2}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}y$).

(2) The only way $q(x,y)$ can equal 0 is if $a = 0$ or if $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}y$. But this latter condition is impossible if $b^2 - 4ac < 0$ and $y \neq 0$, since $x/y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ is imaginary while x and y are real, a contradiction. Now, if $y = 0$ the equation $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}y$ would mean that $x = 0$ as well.
In other words, since $b^2 - 4ac < 0$ means that $\sqrt{b^2 - 4ac}$ is imaginary, the only *real* solution to the equation $a(x - \frac{-b+\sqrt{b^2-4ac}}{2}y)(x - \frac{-b-\sqrt{b^2-4ac}}{2}y) = 0$ is $(0,0)$.

(3) If both $a \neq 0$ and $b^2 - 4ac < 0$, the previous step implies that either $q(x,y) > 0$ for all $(x,y) \neq (0,0)$ or $q(x,y) < 0$ for all $(x,y) \neq (0,0)$. This is because a change in sign for $q(x,y)$, on the unit circle $x^2 + y^2 = 1$, would require $q(x,y)$ to be zero somewhere on the unit circle, which it is not.

Since $a > 0$, this means that $q(1,0) = a > 0$. Since q is positive at one point, it is positive everywhere except $(0,0)$. Therefore the “positive-energy criterion" is true.
In other words, we have shown that **if S satisfies the discriminant test, it satisfies the energy test.**