Math 218D Problem Session

Week 12

1. Rules of vector SVD

Which of the following $A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$ are valid singular value decompositions? Why/why not?

a) $A = 1(1,0)(1,0)^T + 3(0,1)(0,1)^T$

b) $A = 4(1,0)(0,1)^T + 3(0,1)(1,0)^T$

c) $A = 3(1,-1)(1,0)^T + 2(1,1)(0,1)^T$

d) $A = -3(1/\sqrt{2}, -1/\sqrt{2}, 0)(1,0)^T + 2(0,0,1)(0,1)^T$

e) $A = 3(-1/\sqrt{2}, 1/\sqrt{2}, 0)(1,0)^T + 2(0,0,1)(0,1)^T$

f) $A = 5(1,0,0)(0,1)^T + 3(0,1,0)(1,0)^T + 2(0,0,1)(0,1)^T$

2. **The matrix SVD** Suppose that A is an $m \times n$ matrix of rank r, with SVD $A = U\Sigma V^T$.

a) U is a ____ $\times$ ____ matrix, Σ is a ____ $\times$ ____ matrix, and V is a ____ $\times$ ____ matrix. The matrices U and V are ________ matrices. The first ____ diagonal entries of Σ are > 0.

b) Expand A^TA using $A = U\Sigma V^T$ to see that the matrix A^TA has symmetric diagonalization $Q_1D_1Q_1^T$, with $Q_1 =$ ____ and $D_1 =$ ____. What are the eigenvectors of A^TA? What are the eigenvalues?

c) Expand AA^T using $A = U\Sigma V^T$ to see that the matrix AA^T has symmetric diagonalization $Q_2D_2Q_2^T$, with $Q_2 =$ ____ and $D_2 =$ ____. What are the eigenvectors of AA^T? What are the eigenvalues?

d) Suppose that $i \leq r$. The left singular vector u_i is the ith column of U, the singular value σ_i is the ith diagonal entry of Σ, and the right singular vector v_i is the ith column of V. Explain why $Av_i = \sigma_i u_i$ by computing $V^T v_i, \Sigma V^T v_i$, and $U\Sigma V^T v_i$.

3. Computing the vector SVD

To find the vector SVD of a matrix A:

(1) Find the non-zero eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_r > 0$ of $A^T A$.

(2) Find a orthonormal basis of each of the λ_i eigenspace of $A^T A$. Listed in order of decreasing eigenvalue, these are the right singular vectors $v_1, \cdots, v_r$.

(3) For $i = 1, \ldots, r$, set $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \frac{Av_i}{\sigma_i}$. These are the singular values and left singular vectors.

(4) Write $A = \sigma_1 u_1 v_1^T + \cdots + \sigma_r u_r v_r^T$.

Compute the vector SVD of each of the following matrices:

a) $A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

4. Computing the matrix SVD

To find the matrix SVD $A = U\Sigma V^T$ of a matrix A:

(1) Find the symmetric diagonalization VDV^T of A^TA, where the eigenvalues are listed in decreasing order: $\lambda_1 \geq \ldots \geq \lambda_n$. The rank r of A is the same as the number of non-zero eigenvalues of A^TA (counted with multiplicity).

(2) The columns of V are the right singular vectors $v_1, \ldots, v_r$, followed by an orthonormal basis $v_{r+1}, \ldots, v_n$ of Nul(A).

(3) For $i = 1, \ldots, r$, set $\sigma_i = \sqrt{\lambda_i}$ and $u_i = \frac{Av_i}{\sigma_i}$. These are the singular values and left singular vectors.

(4) We still need the vectors $u_{r+1}, \ldots, u_m$: find these by computing an orthonormal basis of Nul(A^T) (using RREF to find a basis, Gram–Schmidt to replace it with an orthonormal basis).

(5) Finally, the matrix U is the matrix with columns $u_1, \ldots, u_m$, the matrix V was found in (1), and Σ has its first r diagonal entries as $\sigma_1, \ldots, \sigma_r$ and the remaining entries of Σ being zero.

Compute the matrix SVD of each of the following matrices:

a) $A = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{pmatrix}$

c) $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 2 & 4 \end{pmatrix}$

5. Sums of rank 1 matrices

This final problem is not about SVDs, but just about sums of rank one matrices.

a) Without computing A, explain why

$$A = (1,2,1)(1,1)^T + (1,-1,1)(-1,1)^T$$

is a rank 2 matrix.

Hint: compute $A(1,1)$ and $A(-1,1)$, and use this to show that $(1,2,1)$ and $(1,-1,1)$ are in the column space of A.

b) If $A = u_1 v_1^T + \cdots + u_r v_r^T$ for some vectors $u_i \in \mathbf{R}^m$ and $v_j \in \mathbf{R}^n$, explain why the rank of A is at most r.

Hint: Show that the subspace $\mathrm{Col}(A)$ is contained in the span $\mathrm{Span}\{u_1, \ldots, u_r\}$, which is at most r-dimensional.

c) If the vectors $u_1, \ldots, u_r \in \mathbf{R}^m$ are a linearly independent set of vectors, and the vectors $v_1, \ldots, v_r \in \mathbf{R}^n$ are also linearly independent, prove that

$$A = u_1 v_1^T + \cdots + u_r v_r^T$$

has rank equal to r.

Hint: Show that there is a vector $v \in \mathbf{R}^n$ which is orthogonal to $v_2, \ldots, v_r$, but $v_1^T v \neq 0$. Compute Av, and use this to show that $u_1 \in \mathrm{Col}(A)$. The same idea shows that $u_2, \ldots, u_r$ are all in $\mathrm{Col}(A)$.