Math 218D Problem Session

Week 2

1. **Solving $Ax=b$ using $PA=LU$**

a) $A=\begin{pmatrix}1&1&1\\2&2&3\\1&2&3\end{pmatrix}$, $b=\begin{pmatrix}1\\3\\2\end{pmatrix}$, $P=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$, $L=\begin{pmatrix}1&0&0\\1&1&0\\2&0&1\end{pmatrix}$, $U=\begin{pmatrix}1&1&1\\0&1&2\\0&0&1\end{pmatrix}$.

b) $Pb=\begin{pmatrix}1\\2\\3\end{pmatrix}$. P swaps the second and third rows of b.

c)

$$\begin{aligned} c_1 \phantom{{}+0c_2+c_3} &= 1 \\ c_1 + c_2 \phantom{{}+c_3} &= 2 \\ 2c_1 + 0c_2 + c_3 &= 3 \end{aligned},$$

and substitution gives $(c_1,c_2,c_3)=(1,1,1)$.

d)

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 1 \\ x_3 &= 1 \end{aligned},$$

and substitution gives $(x_1,x_2,x_3)=(1,-1,1)$.

e) Check $\begin{pmatrix}1&1&1\\2&2&3\\1&2&3\end{pmatrix}\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\3\\2\end{pmatrix}$.

2. Finding $A = LU$ and A^{-1} using elementary matrices

a) $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 4 \\ 1 & 4 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$

The three row operations, in order, are $R_2 -= 2R_1$, $R_3 -= R_1$, $R_3 -= 5R_2$.

We have computed $U = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

b) The elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}.$$

Therefore

$$U = E_3 E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

c)

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} U.$$

d)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}.$$

e)

	L	U
Start	$\begin{pmatrix} (?) & (?) & (?) \\ (?) & (?) & (?) \\ (?) & (?) & (?) \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 4 \\ 1 & 4 & 6 \end{pmatrix}$
Eliminate in C_1	$\begin{pmatrix} 1 & (?) & (?) \\ 2 & (?) & (?) \\ 1 & (?) & (?) \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 4 \end{pmatrix}$
Eliminate in C_2	$\begin{pmatrix} 1 & 0 & (?) \\ 2 & 1 & (?) \\ 1 & 5 & (?) \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$
Eliminate in C_3	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

f)

$$U = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The three row operations are $R_3 \times 1/4$, $R_1 -= 2R_3$, $R_1 += R_2$, corresponding to the elementary matrices

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix},\ E_5 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using these matrices,

$$E_6E_5E_4U = I_3.$$

g)

$$A^{-1} = E_6E_5E_4E_3E_2E_1.$$

h)

$$\left(A \mid I_3\right) = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & 4 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 9 & -5 & 1 \end{array}\right) = \left(U \mid E_3E_2E_1\right).$$

We are halfway done - the right half of this matrix is now $E_3E_2E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 9 & -5 & 1 \end{pmatrix}$.

An aside: one convenient fact about $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is the way in which its entries precisely correspond to the row operations performed. It is harder to interpret the entries of $E_3E_2E_1$. For example, why does 9 appear in $E_3E_2E_1$? It is because

$$\text{final } R_3 = 9(\text{original } R_1) - 5(\text{original } R_2) + (\text{original } R_3).$$

Continuing onwards:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 9 & -5 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9/4 & -5/4 & 1/4 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -7/2 & -5/2 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9/4 & -5/4 & 1/4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9/4 & -5/4 & 1/4 \end{array}\right) = \left(I_3 \mid A^{-1}\right).$$

We conclude that $A^{-1} = \begin{pmatrix} -11/2 & -3/2 & -1/2 \\ -2 & 1 & 0 \\ 9/4 & -5/4 & 1/4 \end{pmatrix}$.