Math 218D Problem Session

Week 3

1. Making some examples

a) Find a matrix with full row rank but not full column rank.

b) Find a matrix with full column rank but not full row rank.

c) Find a matrix with full row rank and full column rank.

For the following examples, it will be helpful to remember that

$$b = Ax = x_1(\text{1st col. of A}) + \cdots + x_n(\text{nth col. of A}).$$

This means that $Ax = b$ is consistent precisely when you can find a way to add (some of) the columns of A to get the vector b.

d) Find a 3×2 matrix A so that $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent. Can you find an example where A has 1 pivot? 2 pivots?

e) Find a 3×2 matrix A so that both $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $Ax = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are consistent. Once you've done this, find a third vector b which makes $Ax = b$ *in*consistent.

f) Find a 2×3 matrix A so that $Ax = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is consistent but $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not consistent.

g) Can you make a 2×3 matrix *with full column rank* (a pivot in each column) so that $Ax = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is consistent but $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not? What about *with full row rank* (a pivot in each row)?

2. Parametric forms

Consider the augmented matrix $\left(A \mid b\right) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -2 & -1 & 1 \\ 0 & -3 & -2 & 1 \end{array}\right)$.

a) Compute the RREF, and verify that the system of equations $Ax = b$ has one free variable. Which variable is it?

b) Find the parametric form of the solution:

$$\begin{aligned} x_1 &= (?) \\ x_2 &= (?) \\ x_3 &= (?) \end{aligned}$$

where all the (?) only involve scalars and the free variable. What are two different solutions to the system of equations?

c) Find the parametric vector form:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\text{free variable}) \cdot \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix} + \begin{pmatrix} (?) \\ (?) \\ (?) \end{pmatrix}$$

where all the (?) are scalars.

d) This is the parametric vector form of a line. Find a point that this line passes through. What direction is the line pointing? Check your answer with this demo.

e) Let's modify the b-vector: $b = (0, 1, -1)$. How many solutions does this new system of equations have? Check your answer geometrically by moving the b vector in the demo linked above.

f) Now find the parametric vector form of the homogeneous equation $Ax = 0$. How is this related to your answer in **d)**? Check your answer geometrically by moving the b vector in the demo linked above.

g) Describe all the b_1, b_2, b_3 which make

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & -2 & -1 & b_2 \\ 0 & -3 & -2 & b_3 \end{array}\right)$$

consistent. Your answer should involve a single linear equation in the variables b_1, b_2, b_3.

h) What shape (point, line, plane, ...) do you get if you add together the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

in every way possible (i.e. take their *span*)? Does the span contain $(0, 1, 1)$? How about $(0, 1, -1)$? or $(0, 0, 0)$? How does this relate to **g)**? Check your answer with this demo.

Hint: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -3 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$

3. The geometry of spans

a) Is it possible to find scalars x_1, x_2 so that

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}?$$

Solve the system algebraically, then geometrically using this demo.

b) Describe

$$\text{Span}\left\{\begin{pmatrix}1\\-1\\5\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}.$$

Is it a point, line, plane, or all of $\mathbf{R}^3$? How do you know? Check your answer with this demo.

c) Find an equation $ax+by+cz=d$ (a, b, c, d scalars) for the plane parametrized by

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

as we vary x_1 and x_2.

Hint: Describe all the vectors $b=(b_1,b_2,b_3)$ which make

$$\begin{pmatrix}1&1\\-1&-1\\5&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

consistent.

d) It is possible to find scalars x_1, x_2 so that

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}4\\-4\\0\end{pmatrix}?$$

Explain why, *without* finding x_1 and x_2. Then find x_1 and x_2 using this demo.

e) Describe the span of the vectors $\begin{pmatrix}1\\-1\\5\end{pmatrix},\begin{pmatrix}1\\-1\\1\end{pmatrix},\begin{pmatrix}4\\-4\\0\end{pmatrix}$. Is it a point, line, plane, or all of $\mathbf{R}^3$? How do you know? Check your answer with this demo.