Math 218D Problem Session

Week 3

1. Making some examples

a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}$ has 1 pivot. $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & 0 \end{pmatrix}$ has 2 pivots.

e) $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}$. $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ would work, as would any vector b such that

$$b_1 - 2b_2 + b_3 \neq 0.$$

f) $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

g) There is no 2×3 matrix with full column rank. Any 2×3 matrix with full row rank (2 pivots) is always consistent, so we can't find one such that $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is inconsistent.

2. Parametric forms

a) The RREF is $\left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

b) The parametric form of the solution is:

$$\begin{aligned} x_1 &= -\tfrac{1}{3}x_3 + \tfrac{1}{3} \\ x_2 &= -\tfrac{2}{3}x_2 - \tfrac{1}{3} \\ x_3 &= \ x_3 \end{aligned} .$$

Setting $x_3 = 0$ gives one solution: $(x_1, x_2, x_3) = (1/3, -1/3, 0)$. Setting $x_3 = 1$ gives another solution $(x_1, x_2, x_3 = (0, -1, 1)$.

c) The parametric vector form is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}.$$

d) The line passes through the point $(1/3, -1/3, 0)$, and goes in the direction of the vector $(-1/3, -2/3, 1)$.

e) This system of equations has no solutions.

f) The parametric vector form of this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix}.$$

The solution to the homogeneous equation is a line, parallel to the line from part d), passing through the origin.

g) A vector $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ makes $Ax = b$ consistent precisely when $-b_1 + b_2 - b_3 = 0$.

h) The span of these vectors is the same as the set of vectors making $Ax = b$ consistent. By g), this is the same as the vectors which satisfying a single linear equation. The set of vectors satisfying a single linear equation is a plane.

3. The geometry of spans

a) No, it is not possible. You can confirm this by computing the RREF of $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 5 & 1 & 0 \end{array}\right)$.

Alternately, you could observe that that the first two components of $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ add up to 0, while the first two components of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ do not.

b) It is all of $\mathbf{R}^3$, since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is not contained in the plane $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$ (using 3a)).

c) By computing the REF of $\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & -1 & b_2 \\ 5 & 1 & b_3 \end{array}\right)$, we confirm that the vectors $b = (b_1, b_2, b_3)$ which make

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

consistent are precisely those where $b_1 + b_2 = 0$. This means that the plane parametrized by

$$x_1\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

has equation

$$x + y = 0.$$

d) Yes, you can find scalars x_1, x_2 so that

$$x_1\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},$$

since $(4, -4, 0)$ solves the equation $x + y = 0$ found in 3c).

e) The vectors $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are not parallel, so they span a plane. The third vector $\begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$ is contained in that plane by 3d), so adding it to the list of vectors does not enlarge the span.