Math 218D Problem Session

Week 4

1. Subspaces?

Decide if each of the following sets of vectors is or is not a subspace, and explain why or why not.

a)
$$\{(x,y,z)\in \mathbf{R}^3 : x+y=1-z\}$$

b)
$$\{(x,y)\in \mathbf{R}^2 : x-2y=0\}$$

c) For A a 3×3 matrix, the set
$$\left\{ v\in \mathbf{R}^3 : Av=\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

d)
$$\left\{(x,y)\in \mathbf{R}^2 : \begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ -1 & -2 & -3\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\end{pmatrix}\right\}$$

e)
$$\{(x,y)\in \mathbf{R}^2 : x^2+y^2=1\}$$

f)
$$\{(x,y)\in \mathbf{R}^2 : x^2+2xy+y^2=0\}$$

The 4 fundamental subspaces associated to a matrix A are the null space, the column space, the row space, and the left-null space.

(1) The null space is the solution set of $Ax = 0$.
(2) The column space is the span of the columns of A.
(3) The row space is the span of the rows of A.
(4) The left-null space is the solution set of $A^T x = 0$.

2. The fundamental subspaces I

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

a) Describe the four fundamental subspaces of the matrix A: are they points? lines? planes?

Hint: The null space Nul(A) has the same dimension as the number of free variables of A. Similarly, the left-null space Nul(A^T) has the same dimension as the number of free variables of A^T.

b) Compute dim(Nul(A))+dim(Row(A)), where dim refers to the *dimension* of the subspace. The dimension of a point, line, or plane is 0, 1, or 2, respectively.

3. The fundamental subspaces II

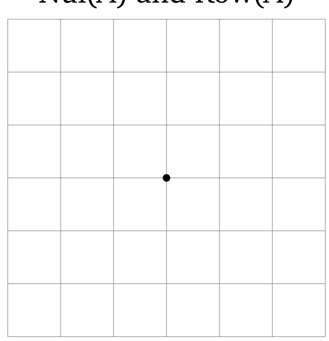
$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

a) Find spanning sets for each of the four fundamental subspaces of the matrix A.

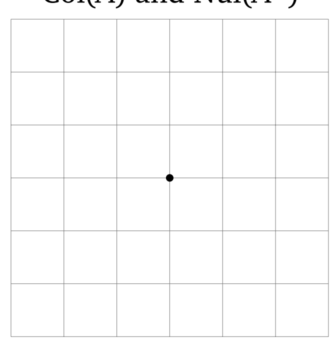
Hint: A spanning set for the null space Nul(A) can be found by finding the parametric vector form of the solution set of $Ax = 0$.

b) Draw each of the fundamental subspaces:

Row Picture
Nul(A) and Row(A)



Column Picture
Col(A) and Nul(A^T)



c) Compute dim(Nul(A)) + dim(Row(A)).

d) Describe a geometric relationship between Nul(A) and Row(A). Then describe the relationship between Col(A) and Nul(A^T).

4. The fundamental subspaces III

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Is the row space Row(A) a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

b) Is the null space Nul(A) a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

c) Is the column space Col(A) a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

d) Is the left-null space Nul(A^T) a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

e) Two of the four subspaces are contained in $\mathbf{R}^2$. For this matrix, each of them is a line. Draw these two subspaces, and describe the geometric relationship between them.

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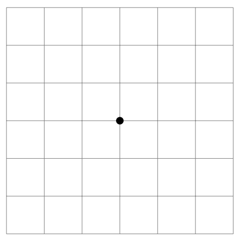

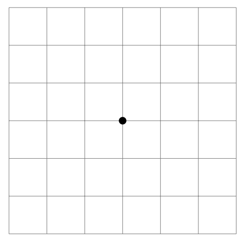

f) Two of the four subspaces are contained in $\mathbf{R}^3$. For this matrix, one is a line and the other is a plane. Determine which is which.

g) Find a vector whose span is the line.

h) Find two vectors whose span is the plane.

i) Find an equation $a_1x + a_2y + a_3z = 0$ for the plane.

Hint: Make the two vectors from h) into the columns of a matrix B. Find an equation which b_1, b_2, b_3 must satisfy in order for $B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to be consistent.

j) What can you observe about the relationship between the answers to **g)** and **i)**? What does this mean geometrically?