Math 218D Problem Session

Week 4

1. Subspaces?

a) Not a subspace, since it doesn't contain $(0,0,0)$.

b) A subspace, since it is the solution set of a homogeneous linear equation.

c) Not a subspace, since it doesn't contain $(0,0,0)$.

d) A subspace, since it is the left-null space of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$

e) Not a subspace, since it doesn't contain $(0,0,0)$.

f) A subspace, since

$$\{(x,y)\in \mathbf{R}^2 : x^2+2xy+y^2=0\} = \{(x,y)\in \mathbf{R}^2 : (x+y)^2=0\} = \{(x,y)\in \mathbf{R}^2 : x+y=0\}.$$

2. The fundamental subspaces I

a) The null space and left null space are points, while the row and column spaces are all of $\mathbf{R}^2$.

b) $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Row}(A)) = 2$

3. The fundamental subspaces II

a) The spanning sets are $\mathrm{Row}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$, $\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\-2\end{pmatrix}\right\}$, $\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}$, $\mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}$, although other answers are possible.

b) Draw the lines spanned by the vectors of **a)**.

c) $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Row}(A)) = 2$.

d) The lines $\mathrm{Nul}(A)$ and $\mathrm{Row}(A)$ are perpendicular. The lines $\mathrm{Col}(A)$ and $\mathrm{Nul}(A^T)$ are perpendicular.

4. The fundamental subspaces III

a) The row space $\mathrm{Row}(A)$ is a subspace of $\mathbf{R}^3$

b) The null space $\mathrm{Nul}(A)$ is a subspace of $\mathbf{R}^3$

c) The column space $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^2$

d) The left-null space $\mathrm{Nul}(A^T)$ is a subspace of $\mathbf{R}^2$

e) The column space is the line spanned by $\begin{pmatrix}1\\-2\end{pmatrix}$, and the left-null space is the line spanned by $\begin{pmatrix}2\\1\end{pmatrix}$.

f) The row space is spanned by the vectors $(1,-1,2)$ and $(-2,2,-4)$, but these are scalar multiples of each other, so the row space is a line. The null space can be found via RREF: $\text{rref}(A) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. The free variables are y and z, and the parametric form is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Therefore the null space is a plane in $\mathbf{R}^3$.

g) $\text{Row}(A) = \text{Span}\{(1,-1,2)\}$

h) $\text{Nul}(A) = \text{Span}\{(1,1,0),(-2,0,1)\}$

i) We consider the matrix $B = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, whose column space equals $\text{Nul}(A)$. We find an equation for the column space of B by finding the REF of $\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 1 & 0 & b_2 \\ 0 & 1 & b_3 \end{array}\right)$, and finding the equation which makes the system consistent. The REF of this augmented matrix is

$$\left(\begin{array}{cc|r} 1 & -2 & b_1 \\ 0 & 2 & b_2 - b_1 \\ 0 & 0 & b_1 - b_2 + 2b_3 \end{array}\right).$$

The equation that (b_1, b_2, b_3) must satisfy to be in the column space of B (and hence the null space of A) is $b_1 - b_2 + 2b_3 = 0$. In other words, the equation for the plane $\text{Nul}(A)$ is

$$x - y + 2z = 0.$$

j) The coefficients of the equation are $(1,-1,2)$. This is the same as the vector which spanned $\text{Row}(A)$ (you may have gotten a scalar multiple of the vector spanning $\text{Row}(A)$ instead.) The means that every vector in the plane is perpendicular to the vector $(1,-1,2)$, i.e. that the plane has *normal vector* $(1,-1,2)$. In other words, *the null space is orthogonal to the row space*. We will discuss the orthogonality of subspaces in more detail in Week 6.