Math 218D Problem Session

Week 5

1. Linear (in)dependence

a) Are the vectors $\binom{1}{0}$, $\binom{1}{-1}$ linearly independent? If not, write down a linear dependence relation.

b) Are the vectors $\binom{1}{1}$, $\binom{1}{-1}$, $\binom{3}{2}$ linearly independent? If not, write down a linear dependence relation.

c) What is the dimension of $\operatorname{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$? Why?

d) Consider 2 linearly independent vectors $u, v \in \mathbf{R}^n$. Show that the two vectors $u+v$, $u-v$ are linearly independent.

e) Consider 3 vectors $u, v, w \in \mathbf{R}^n$. Show that the three vectors $u+v$, $u+2v-w$, $v-w$ are linearly *dependent*.

f) Show that the vectors

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix}, \begin{pmatrix}2\\3\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

are linearly dependent, by writing down a linear dependence relation among them.

Hint: Write down the matrix A whose columns are these vectors, and find a non-zero vector in $\operatorname{Nul}(A)$. Why does this solve the question?

2. Bases from an LU decomposition

Suppose that you have an $A = LU$ decomposition, where

$$U = \begin{pmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but you don't know L or A.

a) Which of the subspaces $\mathrm{Row}(A)$, $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Nul}(A^T)$ can you find a basis for? If you *can* find a basis, do.

b) Which of the subspaces $\mathrm{Row}(A)$, $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Nul}(A^T)$ can you find the dimension of? If you *can* find a dimension, do.

3. Computing all of the bases at once

We will find a basis for each of the four fundamental subspaces of the 7×10 matrix

$$A = \begin{pmatrix} -2 & -2 & 1 & 2 & -4 & -4 & 1 & -8 & 9 & 1 \\ -4 & -3 & 0 & 0 & 11 & -5 & -9 & 5 & -5 & 3 \\ -5 & -5 & 1 & 3 & -1 & -8 & -3 & -10 & 8 & 6 \\ 3 & 3 & -1 & -2 & 2 & 5 & 0 & 9 & -9 & -4 \\ 4 & 4 & 0 & 1 & -11 & 4 & 8 & -4 & 7 & -10 \\ 2 & 2 & 0 & -3 & 5 & 4 & 1 & 6 & -3 & 1 \\ 3 & 3 & 0 & -3 & 3 & 6 & 2 & 8 & -5 & -1 \end{pmatrix}$$

by having a computer do Gauss–Jordan elimination one time. To do this, we will compute the RREF of the matrix $(A \mid I_7)$. The RREF is of the form $(U \mid E)$, and $EA = U$.

Load up linalg.js in your browser, and open a Javascript console. Copy this code to initialize the matrix:

```
A = mat(
    [-2,-2, 1, 2, -4,-4, 1, -8, 9,  1],
    [-4,-3, 0, 0, 11,-5,-9,  5,-5,  3],
    [-5,-5, 1, 3, -1,-8,-3,-10, 8,  6],
    [ 3, 3,-1,-2,  2, 5, 0,  9,-9, -4],
    [ 4, 4, 0, 1,-11, 4, 8, -4, 7,-10],
    [ 2, 2, 0,-3,  5, 4, 1,  6,-3,  1],
    [ 3, 3, 0,-3,  3, 6, 2,  8,-5, -1])
```

Now augment by the identity:

```
  // Create a blank 7x(10+7) matrix
Aug = Matrix.zero(7, 10+7);
  // Put A in the left half
Aug.insertSubmatrix(0, 0, A);
  // Augment with 7x7 identity matrix
Aug.insertSubmatrix(0, 10, Matrix.identity(7));
  // Check that you got the right thing
console.log(Aug.toString(0));
```

Finally, compute the RREF:

```
console.log(Aug.rref().toString(1));
```

a) The non-zero rows of U form a basis for Row(A). What are they?

b) The columns of U with pivots are *not* a basis of Col(A). However, the corresponding columns of A do form a basis for Col(A). What is a basis of Col(A)?

c) You can use U (the RREF of A) to find a basis of Nul(A) = Nul(U). What is a basis of Nul(A)?

d) The bottom row of U is all zero. Look at the bottom row of E. This row forms a basis of Nul(A^T) — what is it?

4. Full row/column rank

Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "A has full column rank"?

a) $\text{Nul}(A) = \{0\}$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \text{Row}(A) = n$

e) The columns of A span $\mathbf{R}^m$

f) A^T has full column rank

Which of the following are equivalent to the statement "A has full row rank"?

a) $\text{Col}(A) = \mathbf{R}^m$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \text{Nul}(A) = n - m$

e) The rows of A span $\mathbf{R}^n$

f) A^T has full column rank