Math 218D Problem Session

Week 5

1. Linear (in)dependence

a) Since neither vector is a scalar multiple of the other, the two vectors are linearly independent.

b) Any 3 vectors in $\mathbf{R}^2$ must be linearly dependent. To find a dependence, we will compute the null space of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}$. Since the RREF of A is $\begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & 1/2 \end{pmatrix}$, we find that $\begin{pmatrix} -5/2 \\ -1/2 \\ 1 \end{pmatrix}$ a vector in the null space. In other words, $-\frac{5}{2}\binom{1}{1} - \frac{1}{2}\binom{1}{-1} + \binom{3}{2} = 0$ is a linear dependence relation among these three vectors.

c) The dimension is the same as the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, since the rank of a matrix equals the dimension of its column space. The rank of this matix is 3 (you could compute its REF, or notice that the transpose A^T is already in REF).

d) Consider any two scalars a, b such that

$$a(u+v) + b(u-v) = 0.$$

We need to show that both of these scalars are in fact equal to 0 - this would show that no linear dependence relations between $u+v$ and $u-v$ are possible.

The first equation implies that $(a+b)u + (b-a)v = a(u+v) + b(u-v) = 0$. Since u and v are linearly independent, this implies that $a+b = 0$ and $b-a = 0$. You can solve these two equations to find $a = 0, b = 0$.

e) The vectors $u+v$, $u+2v-w$, $v-w$ are linearly dependent, since $(u+v)+(v-w) = u+2v-w$.

f) The matrix $A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & -4 & 1 & 1 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The columns of the RREF are dependent: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 0 \end{pmatrix} = 0$. The same dependence relation works for the original vectors (since RREF doesn't change the null space):

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix} - \begin{pmatrix}2\\3\\0\\1\end{pmatrix} - 4\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = 0.$$

2. Bases from an LU decomposition

Suppose that you have an $A = LU$ decomposition, where

$$U = \begin{pmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but you don't know L or A.

a) We can find a basis for Row(A) and Nul(A) - since row operations change Col(A) and Nul(A^T), we can't hope to find them using U. A basis for Row(A) comes from the non-zero rows of U:

$$(1,-1,2,3,5),\ (0,0,1,2,2),\ (0,0,0,01).$$

To find the null space basis, we finish putting A into RREF – its RREF is $\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The parametric equations for $Ax = 0$ are then

$$\begin{aligned} x_1 &= x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

and the parametric vector form is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. A basis for the null space Nul(A) is given by the two vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

b) We can find all of the dimensions: $\dim \text{Row}(A) = 3$ and $\dim \text{Nul}(A) = 2$ from part a). Since row rank equals column rank equals the number of pivots, $\dim \text{Col}(A) = 3$. Since $\dim \text{Col}(A) + \dim \text{Nul}(A^T) = \#$ of rows $= 4$, we find that $\dim \text{Nul}(A^T) = 1$.

3. Computing all of the bases at once

The matrix A is

$$A = \begin{pmatrix} -2 & -2 & 1 & 2 & -4 & -4 & 1 & -8 & 9 & 1 \\ -4 & -3 & 0 & 0 & 11 & -5 & -9 & 5 & -5 & 3 \\ -5 & -5 & 1 & 3 & -1 & -8 & -3 & -10 & 8 & 6 \\ 3 & 3 & -1 & -2 & 2 & 5 & 0 & 9 & -9 & -4 \\ 4 & 4 & 0 & 1 & -11 & 4 & 8 & -4 & 7 & -10 \\ 2 & 2 & 0 & -3 & 5 & 4 & 1 & 6 & -3 & 1 \\ 3 & 3 & 0 & -3 & 3 & 6 & 2 & 8 & -5 & -1 \end{pmatrix}.$$

The RREF of A is

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & -3 & -2 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix E is

$$E = \begin{pmatrix} 0 & -1 & 12.2 & 12.2 & 2.8 & 4.8 & 0.2 \\ 0 & 1 & -5.4 & -5.4 & -0.6 & -1.6 & -0.4 \\ 0 & 0 & 7.8 & 6.8 & 2.2 & 2.2 & 1.8 \\ 0 & 0 & 0.8 & 0.8 & 0.2 & -0.8 & 0.8 \\ 0 & 0 & -2.2 & -2.2 & -0.8 & -1.8 & 0.8 \\ 0 & 0 & -2.4 & -2.4 & -0.6 & -0.6 & -0.4 \\ 1 & 0 & -2.2 & -1.2 & -0.8 & -0.8 & -0.2 \end{pmatrix}.$$

a) A basis for the row space is the list of vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -5 \\ 0 \\ 0 \\ 2 \\ -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -2 \\ 0 \\ 0 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -3 \\ 0 \\ 0 \\ 0 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 2 \\ 1 \end{pmatrix}.$$

b) The column space has basis given by the 1st, 2nd, 3rd, 4th, 6th, and 7th columns of A (not U).

c) The null space is 9-dimensional. Here is a quicker way to find the basis than rewriting in PVF. Remove the zero rows from U, and add an extra row with a -1 for each column without a pivot:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & -3 & -2 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Then each column whose diagonal entry is -1 is one of the vectors in a null space basis.

d) The vector $(1, 0, -2.2, -1.2, -0.8, -0.8, -0.2)$ is a basis of $\mathrm{Nul}(A^T)$.

4. Full row/column rank

Let A be an $m \times n$ matrix. Which of the following are equivalent to the statement "A has full column rank"?

a) $\mathrm{Nul}(A) = \{0\}$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \mathrm{Row}(A) = n$

e) The columns of A span $\mathbf{R}^m$

f) A^T has full column rank

Answer: Full column rank means that A has rank n, $\mathrm{Nul}(A) = \{0\}$, and $\mathrm{Row}(A) = \mathbf{R}^n$. The non-zero vectors in the null space are the same as linear dependency relations among the columns. The statements **a), c), and d)** are equivalent to "A has full column rank".

Which of the following are equivalent to the statement "A has full row rank"?

a) $\mathrm{Col}(A) = \mathbf{R}^m$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \mathrm{Nul}(A) = n - m$

e) The rows of A span $\mathbf{R}^n$

f) A^T has full column rank

Answer: Full row rank means that A has rank m, $\mathrm{Col}(A) = \mathbf{R}^m$, $\dim \mathrm{Row}(A) = m$. Since $\mathrm{rank}(A^T) = \mathrm{rank}(A)$, the matrix A^T also has full column rank. The statements **a), b), d), f)** are equivalent to "A has full row rank".