Math 218D Problem Session

Week 6

1. Projection onto a line

For each of the following,

(1) project the vector b onto the line $V = \text{Span}\{v\}$;

(2) draw the three vectors $b, b_V, b_{V^\perp}$;

(3) compute the projection matrix $P = \frac{vv^T}{v^Tv}$.

a) $b = (1,1),\ v = (1,0)$

b) $b = (0,2),\ v = (1,1)$

c) $b = (1,2,3),\ v = (1,1,-1)$.

2. Planes and normal vectors

The subspace $V = \text{Span}\{(1,1,2),(1,3,1)\}$ of $\mathbf{R}^3$ is a plane.

a) Make the vectors $(1,1,2),(1,3,1)$ into the rows of a 2×3 matrix A - this means that $\text{Row}(A) = V$. Find a basis for $\text{Nul}(A)$. Since

$$V^\perp = \text{Row}(A)^\perp = \text{Nul}(A),$$

you have found a basis $v = (a,b,c)$ for the line $V^\perp$.

In other words, you have found a basis for $V^\perp$ by solving the two orthogonality equations

$$(a,b,c)\cdot(1,1,2) = a + b + 2c = 0,$$
$$(a,b,c)\cdot(1,3,1) = a + 3b + c = 0.$$

b) Confirm that V is the plane $\{(x,y,z)\in\mathbf{R}^3 : ax + by + cz = 0\}$, by showing that both $(1,1,2)$ and $(1,3,1)$ solve this equation. *The coefficients of a plane's equation make a normal vector for the plane.*

c) Find the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1,1,1)$ with respect to the plane V and the orthogonal line $V^\perp$.

Hint: It is easier to compute $b_{V^\perp}$, as it is the projection of b onto the line $V^\perp$ spanned by the vector $v = (a,b,c)$.

3. Projection onto a plane

Consider the plane

$$V = \text{Span}\{(1,1,1,1),(1,2,3,4)\}$$

in $\mathbf{R}^4$. We will find the orthogonal projection of $b = (1,-1,-3,-5)$ onto V. This is a vector $b_V \in \mathbf{R}^4$ so that $b_V \in V$ and $b_{V^\perp} = b - b_V \in V^\perp$.

Since b_V is in V, it must equal

$$b_V = \widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)$$

for some scalars $\widehat{x}_1$ and $\widehat{x}_2$. **We will compute the orthogonal projection by solving for these scalars.**

The vector $b_{V^\perp}$ is orthogonal to every vector in V, in particular it is orthogonal to both $(1,1,1,1)$ and $(1,2,3,4)$. We get two equations:

$$(1,1,1,1)\cdot b_{V^\perp} = 0,$$
$$(1,2,3,4)\cdot b_{V^\perp} = 0.$$

Expanding $b_{V^\perp} = b - b_V = (1,-1,-3,-5) - (\widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4))$, we can rewrite these two equations as

$$(1,1,1,1)\cdot(\widehat{x}_1(1,1,1,1)+\widehat{x}_2(1,2,3,4)) = (1,1,1,1)\cdot(1,-1,-3,-5),$$
$$(1,2,3,4)\cdot(\widehat{x}_1(1,1,1,1)+\widehat{x}_2(1,2,3,4)) = (1,2,3,4)\cdot(1,-1,-3,-5).$$

a) By computing the dot-products, convert this into two linear equations in the two unknowns $\widehat{x}_1$ and $\widehat{x}_2$.

b) Solve for $\widehat{x}_1$ and $\widehat{x}_2$, and compute the orthogonal projection

$$b_V = \widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4).$$

c) Confirm that the vector $b_{V^\perp} = b - b_V$ is orthogonal to V by checking that

$$b_{V^\perp}\cdot(1,1,1,1) = 0 \text{ and } b_{V^\perp}\cdot(1,2,3,4) = 0.$$

d) Write down a matrix A whose column are the two vectors which span V, and compute A^TA, the "matrix of dot products". Compute the vector A^Tb. Explain where the matrix equation $A^TA\widehat{x} = A^Tb$ (the **normal equation**) appears in **a)-b)**, and also where the product $b_V = A\widehat{x}$ appears.

e) Compute the projection matrix $P = A(A^TA)^{-1}A^T$ for the subspace V.

f) Compute the vectors $(I_4 - P)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $(I_4 - P)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$. Explain why these two vectors give a basis for the plane $V^\perp$.

g) Use your answer to **f)** to describe the plane V via *two* implicit equations:

$$V = \{(x_1,x_2,x_3,x_4)\in\mathbf{R}^4 : c_1x_1+c_2x_2+c_3x_3+c_4x_4 = 0 \text{ and } c_1'x_1+c_2'x_2+c_3'x_3+c_4'x_4 = 0\}.$$

In other words, what coefficient vectors (c_1,c_2,c_3,c_4) and (c_1',c_2',c_3',c_4') can we use to describe V, and why? Confirm that every vector in V satisfies these equations by checking that both $(1,1,1,1)$ and $(1,2,3,4)$ do.

4. Some mistakes to avoid

A false “fact”: every projection matrix $P = A(A^TA)^{-1}A^T$ equals the identity matrix I.

A false “proof":

$$P = A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = (AA^{-1})((A^T)^{-1}A^T) = I \cdot I = I.$$

a) What is wrong would this proof?

b) In what case would this proof be correct?

Consider the subspace $V = \text{Span}\{(1,1,1,-1),\ (2,1,1,2),\ (3,2,2,1)\}$ in $\mathbb{R}^4$. V is the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

c) It would be *incorrect* to say that $P = A(A^TA)^{-1}A^T$ is the projection matrix for V. Why?

Hint: Try computing P - what goes wrong?

d) How could you modify A so that $P = A(A^TA)^{-1}A^T$ is the projection matrix for V?