Math 218D Problem Session

Week 7

1. Orthogonal matrices

A *orthogonal matrix* is a *square* matrix Q whose columns form an ortho*normal* set. Alternately, it is a square matrix Q such that $Q^T Q = I_n$.

a) Is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ an orthogonal matrix?

b) Is $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ an orthogonal matrix?

c) Is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ an orthogonal matrix?

2. Rotation and reflection

A *rotation matrix* $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ is an example of an orthogonal matrix.

a) Confirm that R_θ is an orthogonal matrix by checking $R_\theta^T R_\theta = I_2$.

b) Draw the vectors $R_{\pi/6}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $R_{\pi/6}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

c) Using dot products, compute the angle between the rotated vectors $R_{\pi/6}\binom{1}{0}$ and $R_{\pi/6}\binom{1}{1}$. Confirm that this is the same as the angle between the two vectors $\binom{1}{0}$ and $\binom{1}{1}$. **This is an example of a general phenomenon: multiplying by an orthogonal matrix preserves angles and lengths.**

Consider a line $L = \text{Span}\{v\} \subset \mathbf{R}^3$, and the orthogonal complement plane $V = L^\perp$. The *reflection matrix* for *reflection across* V is the orthogonal matrix

$$Q = I_3 - 2P_L,$$

where P_L is the projection matrix for L.

d) When $L = \text{Span}\{(0,1,0)\}$, compute the reflection matrix Q. Draw the line L and the plane V. Compute and draw the vector $(1,1,0)$, the projection $P_L(1,1,0)$, and the reflection $Q(1,1,0)$.

e) Confirm that *any* reflection matrix $Q = I_3 - 2P_L$ is an orthogonal matrix by showing that $Q^T Q = (I_3 - 2P_L)^T(I_3 - 2P_L)$ equals I_3.
Hint: Remember that $P_L^2 = P_L$ and $P_L^T = P_L$.

3. Gram-Schmidt and QR

The purpose of the Gram–Schmidt process is to replace a basis $\{v_1, \cdots, v_k\}$ of a subspace $V \subset \mathbf{R}^n$ with an **orthogonal basis** of V (a basis whose vectors are an orthogonal set).

The vectors $v_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are a basis for a plane $V \subset \mathbf{R}^3$. Set

$$u_1 = v_1, \; u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1.$$

These two vectors are the output of the Gram–Schmidt process.

a) Compute $\frac{u_1}{\|u_1\|}$ and $\frac{u_2}{\|u_2\|}$, and confirm that $\left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}\right\}$ is an orthonormal set of vectors (you need to compute 3 dot products).

b) We can find the QR decomposition of $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}$ by setting

$$Q = \begin{pmatrix} | & | \\ \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \\ | & | \end{pmatrix},$$

a 3×2 matrix. Now, $A = QR$ for some upper-triangular matrix R, and you saw a formula for R in lecture. Here is another way to find R:

$$R = Q^T A.$$

Use this to compute R, and confirm that $A = QR$ by multiplying Q times R.

Note: The method of finding R given in lecture is much faster, as it involves only book-keeping your work from finding Q.

c) Explain why this formula for R worked, i.e. why $A = QR$ had to imply that $Q^T A = R$.

Hint: Multiply both sides of $A = QR$ by Q^T. What does $Q^T Q$ always equal, for a matrix Q with orthonormal columns?

4. Least squares

We want to find the line $y = Cx + D$ which best fits the data points $(1,3),(2,2)$, $(-2,1)$ (in the least-squares sense). If there were a line which was an exact fit, the coefficients C and D would solve the equation

$$A\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}. \text{ where } A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ -2 & 1 \end{pmatrix}.$$

But there is no solution to this, as these 3 data points are not collinear. Instead, we'll find the *least-squares solution* $\widehat{x} = \binom{C}{D}$, i.e. the solution to

$$A^T A\widehat{x} = A^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

a) Compute $A^T A$ and $A^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, and solve for the least-squares solutions $\widehat{x} = \begin{pmatrix} C \\ D \end{pmatrix}$.

b) Plot the data points and the least-squares line $y = Cx + D$.

c) What do the numbers in the vector $A\widehat{x}$ mean?

d) Compute the *error* $\left\| A\widehat{x} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \right\|^2$.

e) You already found the QR decomposition for this matrix A in problem 2. Solve the equation

$$R\widehat{x} = Q^T \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

and confirm that this $\widehat{x}$ is the same vector you found in part **a)**.

5. Another Gram–Schmidt

a) Apply the Gram–Schmidt process to the vectors $v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (0,1,1)$ to obtain an orthogonal set u_1, u_2, u_3.
(Recall that $u_1 = v_1, u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1, u_3 = v_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2$.)

b) Find the QR decomposition of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

c) Consider the vector $b = (1,1,1)$. Since $\{u_1, u_2, u_3\}$ is a basis for $\mathbf{R}^3$, there are scalars x_1, x_2, x_3 such that $b = x_1 u_1 + x_2 u_2 + x_3 u_3$. Solve for these scalars by taking the dot product of this equation with each of u_1, u_2, u_3, giving 3 equations

$$b \cdot u_i = (x_1 u_1 + x_2 u_2 + x_3 u_3) \cdot u_i \text{ for } i = 1,2,3.$$

(These equations simplify dramatically when you compute the dot products.)

d) Explain how you could instead solve for these scalars using the formula $QQ^T = P_{\mathbf{R}^3} = I_3$.

Hint: First, $Q(Q^T b) = b$. Second, $Q \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (a_1 \frac{u_1}{\|u_1\|} + a_2 \frac{u_2}{\|u_2\|} + a_3 \frac{u_3}{\|u_3\|})$.