Math 218D Problem Session

Week 8

1. Some quick determinants

Compute the determinants of the following matrices:

a) $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 10 & 17 \\ 0 & 2 & \pi \\ 0 & 0 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

d) $\begin{pmatrix} 0 & 1 \\ 5 & 0 \end{pmatrix}$ **e)** $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$

g) $\begin{pmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 5 & 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ **h)** $\begin{pmatrix} 2 & 5 \\ 1 & 2 \end{pmatrix}^{20}$

2. Some determinants with variables

a) Compute the determinant of each of $A = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right)$, A^2, A^{-1}, and $A - xI_2$. Find the two values of x so that $\det(A - xI_2) = 0$.

b) Compute the determinant of

$$\begin{pmatrix} 1-x & 1 & 1 \\ 2 & 2-x & 2 \\ 1 & 2 & 3-x \end{pmatrix}.$$

This is a polynomial in the variable x—what degree is the polynomial?

3. More cofactor expansion

a) By repeatedly using the cofactor expansion, compute the determinant of

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 1 \end{pmatrix}.$$

Hint: Use the cofactor expansion along a row or column with many zeros.

b) Compute the determinant of $\begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & -1 & 0 & y \end{pmatrix}$ using cofactor expansions.

c) Use repeated cofactor expansion to show that the determinant of

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{pmatrix}$$

is 0 (every $*$ is an unknown entry).

d) Explain why the matrix of **c)** has linearly dependent columns. Why does this mean it must have determinant equal to 0?

4. Signs of determinants

The *sign* of a number is $+1$ if the number is positive and -1 if it is negative.

a) Draw the vectors $u = (1,-1)$, $v = (2,3)$. Is v clockwise or counterclockwise from u? What is the *sign* of the determinant of $\left(\begin{smallmatrix} 1 & -1 \\ 2 & 3 \end{smallmatrix}\right)$?

b) Draw the vectors $u = (-1,2)$, $v = (1,1)$. Is v clockwise or counterclockwise from u? What is the *sign* of the determinant of $\left(\begin{smallmatrix} -1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$?

c) Are the vectors $u = (0,1,0)$, $v = (1,1,0)$, $w = (1,1,1)$ in right-hand order (RHO)? Here is how you tell. With your right hand, point your index finger in the direction of u, your middle finger in the direction of v, and your thumb in the direction of w. When you point your thumb at your face, the vectors are in RHO if the middle finger is CCW of your index finger. Otherwise, the vectors are in left-hand order.

d) Are the vectors $u = (1,1,0)$, $v = (0,1,0)$, $w = (1,1,1)$ in right-hand order or left-hand order?

e) What is the sign of the determinants of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}?$$

f) What do you think the sign of the 3×3 determinant has to do with right hand order? (You'll verify this more carefully in HW#8.18).

5. A recursion

Consider the $n \times n$ matrix C_n with 1's above and below the diagonal:

$$C_1 = \begin{pmatrix} 0 \end{pmatrix}, \ C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ C_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ C_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \ldots$$

a) Compute the determinants of C_1, C_2, C_3, and C_4.

b) Using cofactor expansion, relate $\det(C_n)$ to $\det(C_{n-1})$ and $\det(C_{n-2})$ (for any $n \geq 2$).

c) Using **b)**, find $\det(C_{10})$.