Math 218D Problem Session

Week 8

1. Some quick determinants

a) $\det\left(\begin{pmatrix}1 & 1 & 1\\ 2 & 2 & 2\\ 1 & 2 & 3\end{pmatrix}\right)=0$ **b)** $\det\left(\begin{pmatrix}1 & 10 & 17\\ 0 & 2 & \pi\\ 0 & 0 & 3\end{pmatrix}\right)=6$ **c)** $\det\left(\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}\right)=5$

d) $\det\left(\begin{pmatrix}0 & 1\\ 5 & 0\end{pmatrix}\right)=-5$ **e)** $\det\left(\begin{pmatrix}0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\right)=1$ **f)** $\det\left(\begin{pmatrix}0 & 0 & 2\\ 3 & 0 & 0\\ 0 & 4 & 0\end{pmatrix}\right)=24$

g) $\det\left(\begin{pmatrix}1 & 0 & 0\\ 7 & 3 & 0\\ 5 & 5 & 1\end{pmatrix}\cdot\begin{pmatrix}2 & 2 & 2\\ 0 & 3 & -1\\ 0 & 0 & 1\end{pmatrix}\right)=18$ **h)** $\det\left(\begin{pmatrix}2 & 5\\ 1 & 2\end{pmatrix}^{20}\right)=(-1)^{20}=1$

2. Some determinants with variables

a) $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. $\det(A)=-2$, $\det(A^2)=\det(A)^2=4, \det(A)^{-1}=-\frac{1}{2}, \det(A-xI_2)=\det\left(\begin{pmatrix}1-x & 2 & 3 & 4-x\end{pmatrix}\right)=x^2-5x-2$. The equation $x^2-5x-2=0$ has solutions $x=\frac{5\pm\sqrt{33}}{2}$.

b) Using Sarrus' scheme, $\det\left(\begin{pmatrix}1-x & 1 & 1\\ 2 & 2-x & 2\\ 1 & 2 & 3-x\end{pmatrix}\right)=(1-x)(2-x)(3-x)-(1-x)2\cdot 2+1\cdot 2\cdot 1-1\cdot 2\cdot(3-x)+1\cdot 2\cdot 2-1\cdot(2-x)\cdot 1.$

Simplifying, which is admittedly tedious, gives $\det\left(\begin{pmatrix}1-x & 1 & 1\\ 2 & 2-x & 2\\ 1 & 2 & 3-x\end{pmatrix}\right)=-x^3+6x^2-4x$. This is a degree 3 polynomial.

3. More cofactor expansion

a) We use cofactor expansion three times:

$$\det\left(\begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & 1 \end{pmatrix}\right) = 3\det\left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & -1 & -2 \end{pmatrix}\right) = 3\cdot 1\cdot\det\left(\begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\right) = -3.$$

b) $\det\left(\begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & -1 & 0 & y \end{pmatrix}\right) = xy.$

c) First,

$$\det\left(\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \end{pmatrix}\right) = *\det\left(\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}\right) - *\det\left(\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}\right).$$

Then,

$$\det\left(\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}\right) = *\det\left(\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}\right) = 0.$$

As the 5×5 determinant is a sum of two 4×4 determinants, and each of these is zero, the determinant is zero.

d) The last three columns of this matrix are contained in the (x_1, x_2)-plane. Three vectors in a plane must be linearly dependent. Any matrix with linearly dependent columns is not invertible, hence has determinant equal 0.

4. Signs of determinants

a) The vector v is counterclockwise from u. The sign of the determinant is $+1$.

b) The vector v is clockwise from u. The sign of the determinant is -1.

c) The vectors $u=(0,1,0)$, $v=(1,1,0)$, $w=(1,1,1)$ are in LHO.

d) The vectors $u=(1,1,0)$, $v=(0,1,0)$, $w=(1,1,1)$ are in RHO.

e) The sign of the determinants of

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

are -1 and $+1$.

f) The sign of a 3×3 determinant is $+1$ if the rows are in RHO, and -1 if the rows are in LHO.

5. A recursion

Consider the $n \times n$ matrix C_n with 1's above and below the diagonal:

$$C_1 = \begin{pmatrix} 0 \end{pmatrix},\ C_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ C_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},\ C_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ \ldots$$

a) The determinants of C_1, C_2, C_3, and C_4 are 0, -1, 0, 1.

b) I'll explain for C_4, but the general pattern is the same. We first do cofactor expansion in the first row:

$$\det(C_4) = \det\left(\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\right) = -1 \cdot \det\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right).$$

We then do cofactor expansion in the first column:

$$-1 \cdot \det\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right) = -1 \det\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = -\det(C_2).$$

In other words, we have removed the first and second rows and columns of our original matrix C_4, and are left with C_2 (and a sign).

c) Since $\det(C_2) = -1$, the equation $\det(C_n) = \det(C_{n-2})$ implies that

$$\det(C_{2k}) = (-1)^{k-1} \det(C_2) = (-1)^k.$$

Therefore $\det(C_{10}) = \det(C_{2\cdot 5}) = (-1)^5 = -1$.