

MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

Name		Duke Email	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1.

[20 points]

Consider

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -2 & -1 & -1 & 2 \\ 2 & 3 & 5 & 4 \\ 6 & 3 & -3 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -2 \\ -8 \\ 4 \\ 0 \end{pmatrix}.$$

- a) Carry out Gaussian reduction with maximal partial pivoting to find a $PA = LU$ decomposition. You should obtain

$$U = \begin{pmatrix} 6 & 3 & -3 & 0 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Please write the row operations you performed.

- b) Write the elementary matrices for the row operations you performed.
c) Solve the equations $Ly = Pb$ and $Ux = y$ to find a solution of $Ax = b$.
d) Briefly explain why step b) is faster than solving $Ax = b$ using Gaussian elimination on the augmented matrix $(A | b)$, once you have a $PA = LU$ decomposition.

Solution.

a)
$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

b)
$$\begin{aligned} R_1 \leftrightarrow R_4: & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & R_2 += \frac{1}{3}R_1: & \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & R_3 -= \frac{1}{3}R_1: & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ R_2 \leftrightarrow R_3: & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & R_4 -= \frac{1}{2}R_2: & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix} & R_4 \leftrightarrow R_3: & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ R_4 -= \frac{1}{2}R_3: & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix} \end{aligned}$$

c)
$$Ly = Pb \implies y = \begin{pmatrix} 0 \\ 4 \\ -4 \\ -6 \end{pmatrix} \quad Ux = y \implies x = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$

- d) Gaussian elimination takes about $\frac{2}{3} \cdot 4^3 \approx 43$ flops, whereas forward- and back-substitution take about $4^2 = 16$ flops.

Problem 2.

[15 points]

a) Compute the inverse of $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}$.

Be sure to write out any row operations you perform.

b) For which value(s) of k is $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & k \\ 2 & 3 & 9 \end{pmatrix}$ not invertible?

c) Suppose that A is a 3×3 matrix whose third column is in the span of the first two. Briefly explain why A is not invertible.

Solution.

a)
$$\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -69 & -27 & 8 \\ -8 & -3 & 1 \\ 18 & 7 & -2 \end{pmatrix}$$

b)
$$k = -\frac{36}{7}$$

c) A relation among the columns leads to a nontrivial solution of $Ax = 0$. But if A is invertible then $Ax = 0$ has exactly one solution, namely, $0 = A^{-1}0$.

Problem 3.

[20 points]

Consider

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 \\ -2 & -6 & 6 & -2 \\ 2 & 6 & 3 & -7 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ -8 \\ -10 \end{pmatrix}.$$

- Find the parametric vector form of the solution set of $Ax = b$. Be sure to write out any row operations you perform.
- Write down two different solutions of $Ax = b$. (Your answer will be two vectors with numbers in them.)
- Find a spanning set for $\text{Nul}(A)$.
- Let $v = (-1, 1, 1, 1)$. Check that $v \in \text{Nul}(A)$, and write v as a linear combination of the spanning vectors you obtained in **d**).
[Hint: what values do the free variables have to take?]

Solution.

a)
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

- b) Choose any values of the free variables. For instance, $(x_2, x_4) = (1, 0)$ and $(0, 1)$ give $(-5, 1, -2, 0)$ and $(0, 0, -1, 1)$, respectively.

c)
$$\text{Nul}(A) = \text{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

- d) One computes $Av = 0$, so $v \in \text{Nul}(A)$. The second (resp. fourth) coordinate of v is the value of x_2 (resp. x_4), so

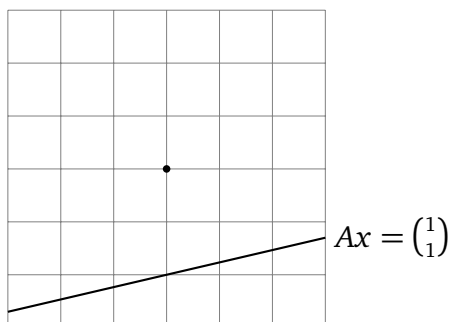
$$v = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 4.

[20 points]

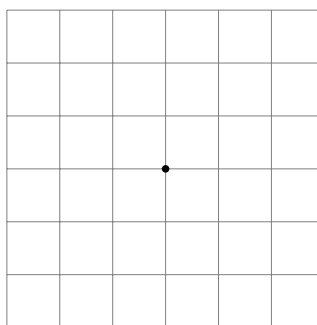
For a certain 2×2 matrix A , the solution set of $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is drawn.

- a) Draw the solution set of $Ax = 0$ and the solution set of $Ax = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ in the grid below. Be sure to label which is which.



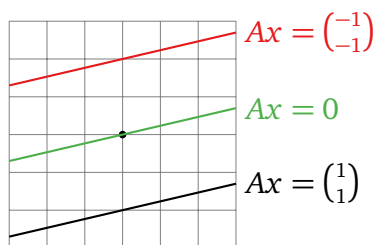
b) $\text{rank}(A) = \boxed{}$

- c) Draw the column space of A . Be precise!



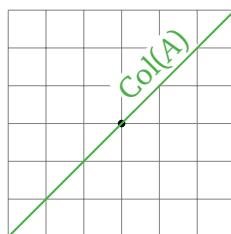
Solution.

a)



- c) The rank is 1, since A has one free variable.

d)



Problem 5.

[15 points]

Find examples of matrices with the following properties. If no such matrix exists, write “no way, man,” or use your favorite colloquialism instead. You need not justify your answers.

- a) A matrix A , in RREF, such that $Ax = b$ has at least one solution for every b , but A does not have a pivot in every column.
- b) A 3×5 matrix of rank 4, in RREF.
- c) A 2×2 matrix A such that the solution set of $Ax = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is a line, and $Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has no solutions.
- d) A 3×3 matrix A with no zero entries, such that $\text{Col}(A)$ is a plane.
- e) A 4×4 matrix A with a pivot in every row such that $A(1, 2, -1, 1) = 0$.

Solution.

- a) There are many answers; one is $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.
- b) Yeah, right.
- c) There are many answers; one is $A = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$.
- d) There are many answers; one is $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.
- e) As if.

Problem 6.

[18 points]

Which of the following are subspaces of \mathbf{R}^4 ? If not, why?

a) $\text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \end{pmatrix} \right\}$

b) $\text{Nul} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

c) $\text{Col} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 0 & 4 \\ 6 & -4 & 2 \\ -9 & 3 & 4 \end{pmatrix}$

d) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

e) $\{\}$

f) $V = \left\{ \text{all vectors } \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \text{ such that } xy = zw \right\}$

Solution.

a) Yes.

b) No: this is a subspace of \mathbf{R}^3 .

c) Yes.

d) Yes.

e) No: this does not contain the zero vector.

f) No: this is not closed under addition. For instance,

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$