MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1.

[20 points]

Consider

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -2 & -1 & -1 & 2 \\ 2 & 3 & 5 & 4 \\ 6 & 3 & -3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ -8 \\ 4 \\ 0 \end{pmatrix}.$$

a) Carry out Gaussian reduction with maximal partial pivoting to find a $PA = LU$ decomposition. You should obtain

$$U = \begin{pmatrix} 6 & 3 & -3 & 0 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Please write the row operations you performed.

$$L = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \qquad P = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

b) Write the elementary matrices for the row operations you performed.

c) Solve the equations $Ly = Pb$ and $Ux = y$ to find a solution of $Ax = b$.

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

d) Briefly explain why step **b)** is faster than solving $Ax = b$ using Gaussian elimination on the augmented matrix $\left(A \mid b\right)$, once you have a $PA = LU$ decomposition.

Problem 2. [15 points]

a) Compute the inverse of $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}$.

Be sure to write out any row operations you perform.

$$\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) For which value(s) of k is $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & k \\ 2 & 3 & 9 \end{pmatrix}$ *not* invertible?

$$k = \boxed{}$$

c) Suppose that A is a 3×3 matrix whose third column is in the span of the first two. Briefly explain why A is not invertible.

Problem 3.

[20 points]

Consider

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 \\ -2 & -6 & 6 & -2 \\ 2 & 6 & 3 & -7 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ -8 \\ -10 \end{pmatrix}.$$

a) Find the parametric vector form of the solution set of $Ax = b$. Be sure to write out any row operations you perform.

$$x = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} +$$

b) Write down two different solutions of $Ax = b$. (Your answer will be two vectors with numbers in them.)

$$x_1 = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \qquad x_2 = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

c) Find a spanning set for Nul(A).

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \qquad \qquad \qquad \right\}$$

d) Let $v = (-1, 1, 1, 1)$. Check that $v \in \mathrm{Nul}(A)$, and write v as a linear combination of the spanning vectors you obtained in **d)**.

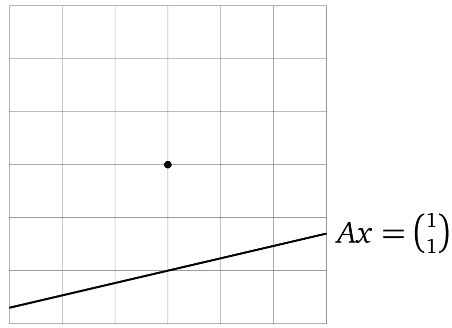
[**Hint:** what values do the free variables have to take?]

Problem 4.

[20 points]

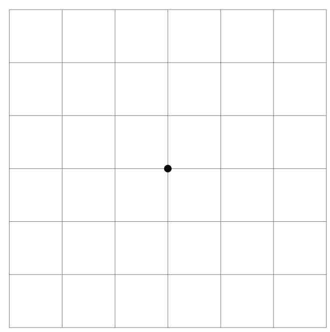
For a certain 2×2 matrix A, the solution set of $Ax = \binom{1}{1}$ is drawn.

a) Draw the solution set of $Ax = 0$ and the solution set of $Ax = \binom{-1}{-1}$ in the grid below. Be sure to label which is which.



b) $\mathrm{rank}(A) = \boxed{}$

c) Draw the column space of A. Be precise!



Problem 5.

[15 points]

Find examples of matrices with the following properties. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A matrix A, in RREF, such that $Ax = b$ has at least one solution for every b, but A does not have a pivot in every column.

b) A 3×5 matrix of rank 4, in RREF.

c) A 2×2 matrix A such that the solution set of $Ax = \binom{3}{4}$ is a line, and $Ax = \binom{1}{1}$ has no solutions.

d) A 3×3 matrix A with no zero entries, such that $\mathrm{Col}(A)$ is a plane.

e) A 4×4 matrix A with a pivot in every row such that $A(1, 2, -1, 1) = 0$.

Problem 6.

[18 points]

Which of the following are subspaces of $\mathbf{R}^4$? If not, why?

a) $\text{Span}\left\{\begin{pmatrix}1\\0\\0\\2\end{pmatrix}, \begin{pmatrix}0\\-2\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\-2\\-1\\2\end{pmatrix}\right\}$

b) $\text{Nul}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

c) $\text{Col}\begin{pmatrix}2 & -1 & 3\\0 & 0 & 4\\6 & -4 & 2\\-9 & 3 & 4\end{pmatrix}$

d) $\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\}$

e) $\{\}$

f) $V = \left\{\text{all vectors } \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \text{ such that } xy = zw\right\}$