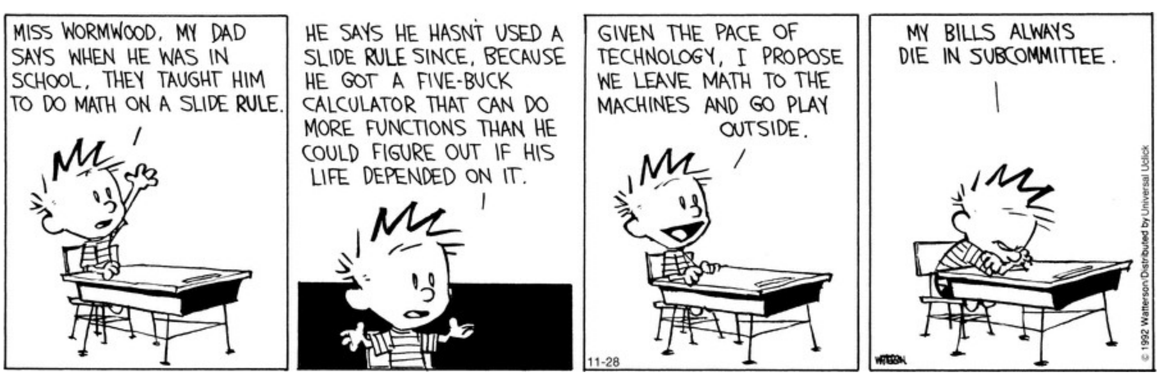
MATH 218D-1
MIDTERM EXAMINATION 1

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



Problem 1.

[25 points]

Consider the matrix

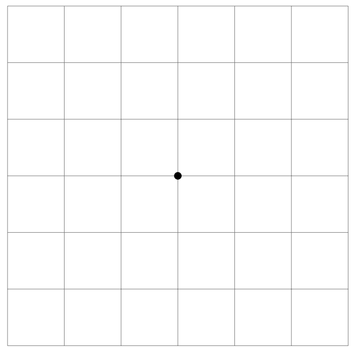
$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Use Gauss–Jordan elimination to put A into reduced row echelon form.

b) The free columns are $\boxed{}$.

c) The rank of A is $\boxed{}$.

d) Draw a picture of the column space Col(A) below.



e) Write down a vector b in $\mathbf{R}^2$ such that $Ax = b$ has no solution. If no such vector exists, explain why not.

f) The null space is a (circle one) $\begin{pmatrix} \text{point} \\ \text{line} \\ \text{plane} \end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\boxed{}}$.

g) Find the general solution of $Ax = \binom{-2}{4}$ in parametric vector form.

h) Express Nul(A) as a span of some number of vectors.

i) Write down any nontrivial solution of $Ax = 0$.

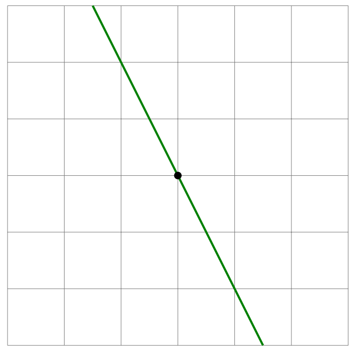
Solution.

a) An REF is $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

b) The free columns are the second and third.

c) The rank is 1.

d)



e) Any b not on this line works. For instance, $b = \binom{1}{1}$.

f) The null space is a plane in $\mathbf{R}^3$.

g)

$$x = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

h)

$$\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

i) For instance, $(1, 1, 0)$.

Problem 2.

[25 points]

Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ -4 & -5 & -3 \\ -2 & -6 & 0 \end{pmatrix}.$$

a) Find a lower-unitriangular matrix L and a matrix U in REF such that $A = LU$. You should end up with

$$U = \begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{pmatrix}.$$

b) Solve the equation $Ax = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ using the LU decomposition you found in **a)**.

c) If you compute a $PA = LU$ decomposition using maximal partial pivoting, what is P? (You do not have to do the bookkeeping to find L in this part.)

d) Find the inverse of A.

Solution.

a)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix}$$

b)

$$x = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

c)

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

d)

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 18 & 6 & 4 \\ -6 & -2 & -2 \\ -14 & -6 & -2 \end{pmatrix}$$

Problem 3.

[20 points]

Consider the subspace V of $\mathbf{R}^4$ defined by the equations

$$x_1 + x_2 = x_3 + x_4$$
$$x_1 + x_3 = x_2 + x_4.$$

a) Express V as the null space of a matrix A.

b) Express V as the span of a set of vectors.

c) Express V as the column space of a (different) matrix B.

d) One of the following vectors is contained in V. Identify which one is contained in V, and express it as a linear combination of the vectors you found in **b)**.

$$\begin{pmatrix}1\\4\\2\\3\end{pmatrix} \qquad \begin{pmatrix}1\\3\\3\\1\end{pmatrix} \qquad \begin{pmatrix}1\\3\\-3\\-1\end{pmatrix}$$

Solution.

a)

$$V = \mathrm{Nul}\begin{pmatrix}1 & 1 & -1 & -1\\1 & -1 & 1 & -1\end{pmatrix}$$

b)

$$V = \mathrm{Span}\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\1\end{pmatrix}\right\}$$

c)

$$V = \mathrm{Col}\begin{pmatrix}0 & 1\\1 & 0\\1 & 0\\0 & 1\end{pmatrix}$$

d)

$$\begin{pmatrix}1\\3\\3\\1\end{pmatrix} = 3\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Problem 4. [15 points]

Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 6 & 7 & 8 \\ 6 & 8 & 10 & 12 \\ -9 & -10 & -11 & -12 \end{pmatrix}.$$

a) What three row operations are needed to transform A into B?

b) What are the elementary matrices E_1, E_2, E_3 for these three operations?

c) Write an equation for B in terms of A and E_1, E_2, E_3.

d) Write an equation for A in terms of B and $E_1^{-1}, E_2^{-1}, E_3^{-1}$.

Solution.

a) First $R_1 \longleftrightarrow R_2$, then $R_2 \mathrel{+}= R_1$, then $R_3 \times= -1$.

b)

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c)

$$B = E_3 E_2 E_1 A$$

d)

$$A = E_1^{-1} E_2^{-1} E_3^{-1} B$$

Problem 5. [10 points]

Consider the subspace

$$V = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}.$$

a) Show that $\begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix}$ is in V.

b) Show that $\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$ is not in V.

c) Circle one: V is a point line plane space.

Solution.

a) We solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix}$$

by row reducing an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 4 & 5 & 6 & -4 \\ 7 & 8 & 9 & -4 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This system has infinitely many solutions, so $(-4,-4,-4)$ is in V.

b) We solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$$

by row reducing an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 4 & 5 & 6 & -4 \\ 7 & 8 & 9 & 4 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This system has no solutions, so $(-4,-4,4)$ is not in V.

c) Since V contains two noncollinear vectors and is not all of $\mathbf{R}^3$, it must be a plane.

Problem 6. [15 points]

Find examples of matrices with the following properties. If no such matrix exists, explain why not.

- **a)** A 2×3 matrix A such that the solution set of $Ax = 0$ is the line spanned by $(1, 2, 1)$ and the solution set of $Ax = \binom{2}{1}$ is the point $\{(1, 0, 0)\}$.
- **b)** A 3×2 matrix A such that $\mathrm{Col}(A) = \mathbf{R}^3$.
- **c)** A 2×3 matrix A such that $\mathrm{Col}(A) = \mathbf{R}^3$.
- **d)** A 2×3 matrix A such that $\mathrm{Nul}(A) = \mathbf{R}^3$.
- **e)** An invertible 2×2 matrix such that $A\binom{1}{2} = A\binom{2}{1}$.

Solution.

- **a)** Impossible: the solution set of $Ax = \binom{2}{1}$ is a translate of the solution set of $Ax = 0$.
- **b)** Impossible: the two columns of A cannot span anything larger than a plane.
- **c)** Impossible: the column space of A lives in $\mathbf{R}^2$.
- **d)** The only example is the zero matrix.
- **e)** Impossible: the equation $Ax = b$ has exactly one solution.