MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [15 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

a) Is $\{v_1, v_2, v_3\}$ linearly independent? If not, find a linear relation.

b) Compute the dimension of $\mathrm{Span}\{v_1, v_2, v_3\}$.

c) Write $(2, 6, -2)$ as a linear combination of v_1, v_2, v_3.

Solution.

a) The vectors are linearly independent.

b) $\mathrm{Span}\{v_1, v_2, v_3\} = \mathbf{R}^3$, which has dimension 3.

c) $\begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} = v_1 + v_2 + 3v_3.$

Problem 2. [20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) $\operatorname{Nul}\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix}$

b) $\operatorname{Col}\begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \\ 4 & -1 & 11 \end{pmatrix}$

c) The subspace of all vectors in $\mathbf{R}^4$ whose entries sum to zero.

d) The line $\{(t, 2t, 3t) : t \in \mathbf{R}\}$.

e) $\mathbf{R}^3$

Solution.

These are the bases you would obtain if you did the problem the same way I did.

a) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \\ 1 \end{pmatrix} \right\}$ **b)** $\left\{ \begin{pmatrix} -3 \\ -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ -9 \\ 0 \\ 1 \end{pmatrix} \right\}$ **c)** $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$ **e)** $\{\}$

Problem 3. [25 points]

In this problem we will consider the best-fit plane $z(x,y) = Bx + Cy + D$ through the data points

$$\begin{pmatrix} 3 \\ -5 \\ b_1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ b_2 \end{pmatrix} \begin{pmatrix} -1 \\ 5 \\ b_3 \end{pmatrix} \begin{pmatrix} 3 \\ -7 \\ b_4 \end{pmatrix}.$$

a) Find the matrix A such that the coefficient vector $\widehat{x} = (B, C, D)$ is the least-squares solution of

$$A \begin{pmatrix} B \\ C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

b) Compute the QR decomposition of A.

c) Given a vector $b = (b_1, b_2, b_3, b_4)$, explain how a computer would quickly compute $\widehat{x}$ using the QR decomposition you found in **b)**.

d) Compute the best-fit plane $z = Bx+Cy+D$ when $(b_1, b_2, b_3, b_4) = (10, -10, 20, -20)$, using your QR decomposition or otherwise.

e) What is the minimum value of

$$\big(z(3,-5) - 10\big)^2 + \big(z(1,1) + 10\big)^2 + \big(z(-1,5) - 20\big)^2 + \big(z(3,-7) + 20\big)^2$$

for all planes $z = Bx + Cy + D$?

Solution.

a)
$$A = \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & 1 \\ 3 & -7 & 1 \end{pmatrix}$$

b)
$$Q = \begin{pmatrix} 3/2\sqrt{5} & 1/2\sqrt{5} & -1/\sqrt{10} \\ 1/2\sqrt{5} & 3/2\sqrt{5} & -1/\sqrt{10} \\ -1/2\sqrt{5} & 3/2\sqrt{5} & 2/\sqrt{10} \\ 3/2\sqrt{5} & -1/2\sqrt{5} & 2/\sqrt{10} \end{pmatrix} \qquad R = \begin{pmatrix} 2\sqrt{5} & -4\sqrt{5} & 3\sqrt{5}/5 \\ 0 & 2\sqrt{5} & 3\sqrt{5}/5 \\ 0 & 0 & \sqrt{10}/5 \end{pmatrix}$$

c) A computer would solve $R\widehat{x} = Q^T b$ by back-substitution.

d) $z = 3x + 3y$

e) This is $\|b - \widehat{b}\|^2 = 640$.

Problem 4.

[20 points]

Consider the subspace V in $\mathbf{R}^4$ defined by $x_1 + 2x_2 - 2x_3 - x_4 = 0$.

a) Compute the orthogonal projection $b_{V^\perp}$ of the vector $b = (0, -3, 3, -2)$ onto $V^\perp$.

b) Compute the orthogonal decomposition $b = b_V + b_{V^\perp}$.

c) Find the matrix P_V for orthogonal projection onto V.

d) Find a basis for $\mathrm{Nul}(P_V)$.

Solution.

$$\textbf{a)}\ b_{V^\perp} = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix} \qquad \textbf{b)}\ b = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix}$$

$$\textbf{c)}\ P = \frac{1}{10}\begin{pmatrix} 9 & -2 & 2 & 1 \\ -2 & 6 & 4 & 2 \\ 2 & 4 & 6 & -2 \\ 1 & 2 & -2 & 9 \end{pmatrix} \qquad \textbf{d)}\ \left\{ \begin{pmatrix} 1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \right\}$$

Problem 5. [16 points]

a) Let A be an $m \times n$ matrix of rank r. Which of the following statements are equivalent to "A has full row rank"?

(1) $\mathrm{Nul}(A^T) = \{0\}$
(2) $n = r$
(3) $\mathrm{Col}(A) = \mathbf{R}^m$
(4) A has linearly independent columns
(5) A has a pivot in every row
(6) A is invertible
(7) $Ax = b$ is consistent for every vector b

b) Explain why the projection matrix P_V onto a subspace V can be written as QQ^T for some matrix Q with orthonormal columns. (What is Q in terms of V?)

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly *de*pendent, but v_3 is not in $\mathrm{Span}\{v_1, v_2\}$. Be sure to label which is v_3.

d) Give an example of a 4×4 matrix A such that $\mathrm{Nul}(A) = \mathrm{Row}(A)$, or explain why no such matrix exists.

Solution.

a) (1),(3),(5),(7)

b) Choose an orthonormal basis for V, then let Q be the matrix with those columns.

c) There are many answers. One is

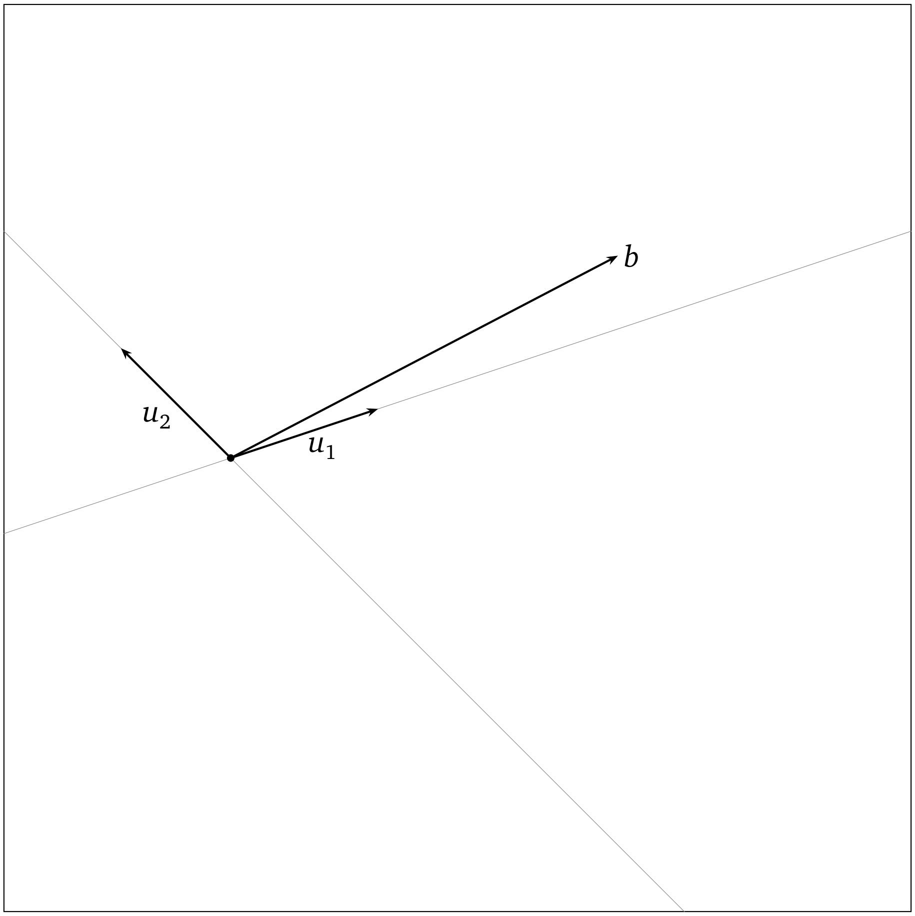
$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

d) This is impossible since $\mathrm{Nul}(A) = \mathrm{Row}(A)^\perp$.

Problem 6.

[10 points]

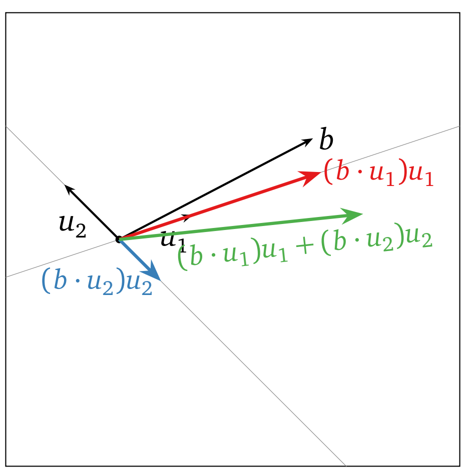
Unit vectors u_1 and u_2 and a vector b are drawn in the picture below. In the same picture, draw and label **a)** $(b \cdot u_1)u_1$ and $(b \cdot u_2)u_2$ and **b)** $(b \cdot u_1)u_1 + (b \cdot u_2)u_2$.



c) Note that $\{u_1, u_2\}$ is a basis for $V = \mathbf{R}^2$. Explain why $b = b_V \neq (b \cdot u_1)u_1 + (b \cdot u_2)u_2$ does not contradict the projection formula.

Solution.

a),b)



c) The vectors u_1, u_2 are not orthogonal.