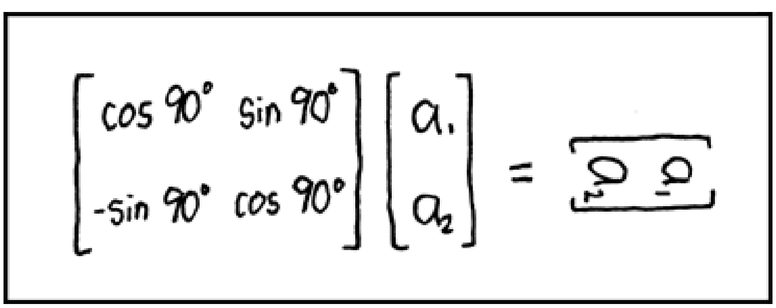
MATH 218D-1
MIDTERM EXAMINATION 2

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



[Hint: this is a joke.]

Problem 1.

[20 points]

Consider the plane

$$V = \{(x, y, z) \ : \ x - y + 2z = 0\}.$$

a) Find a basis for V.

b) Find an orthogonal basis for V.

c) Use the projection formula and your answer to part **b)** to compute the orthogonal projection b_V of the vector $b = (1, 1, -3)$ onto V.

d) Find a basis for $V^\perp$.

e) Find an orthogonal basis of $\mathbf{R}^3$ containing the basis vectors you found in **b)**.

Solution.

a) There are many answers. If you find the solutions of $x - y + 2z = 0$ in parametric vector form, you get

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix} \right\}.$$

b) Running Gram–Schmidt on the above vectors gives

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \right\}.$$

c)
$$b_V = \frac{\begin{pmatrix}1\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{\begin{pmatrix}1\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}}{\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}.$$

d) Since $V = \mathrm{Nul}\begin{pmatrix}1 & -1 & 2\end{pmatrix}$, the orthogonal complement $V^\perp$ is the row space of $\begin{pmatrix}1 & -1 & 2\end{pmatrix}$:

$$\left\{ \begin{pmatrix}1\\-1\\2\end{pmatrix} \right\}.$$

e) We just add the vectors in **b)** and **d)**:

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix} \right\}.$$

(You could also notice that $b - b_V = (-1, 1, -2)$ spans $V^\perp$.)

Problem 2. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}.$$

a) Find the QR decomposition of A. You should get $R = \begin{pmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}$.

b) Solve $R\widehat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$ to find the least-squares solution of $Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$.

c) Compute the matrix P_V for projection onto $V = \mathrm{Col}(A)$.

Solution.

a)

$$Q = \begin{pmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{pmatrix}$$

b)

$$\widehat{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

c)

$$P_V = QQ^T = \frac{1}{10} \begin{pmatrix} 7 & -2 & 3 & 2 & 2 \\ -2 & 2 & 2 & -2 & -2 \\ 3 & 2 & 7 & -2 & -2 \\ 2 & -2 & -2 & 7 & -3 \\ 2 & -2 & -2 & -3 & 7 \end{pmatrix}$$

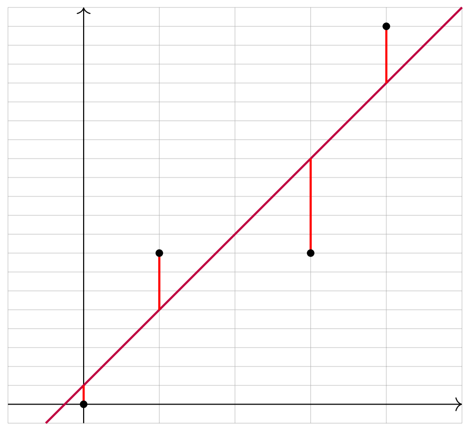
Problem 3.

[15 points]

Consider the data points

$$b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

drawn below.



a) Find the matrix A such that the least-squares solution $\widehat{x} = (C, D)$ of

$$A\begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

gives the coefficients of the best-fit line $y = Cx + D$.

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

c) Compute the minimized vector $b_{V^\perp}$. What does $b_{V^\perp}$ represent in the original best-fit problem? (Here $V = \mathrm{Col}(A)$.)

d) What is the best-fit line among all lines *passing through the origin*?

Solution.

a)

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$$

b)

$$\widehat{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \rightsquigarrow \quad y = 4x + 1$$

c) The minimized vector is

$$b_{V^\perp} = b - A\widehat{x} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}.$$

This is the vector of vertical distances from the data points to the graph of the best-fit line, drawn in red in the picture.

d) Using $y = Cx$ means solving the least-squares problem

$$\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} C = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \quad \rightsquigarrow \quad C = \frac{56}{13}.$$

The best-fit line is $y = \frac{56}{13}x$.

Problem 4.

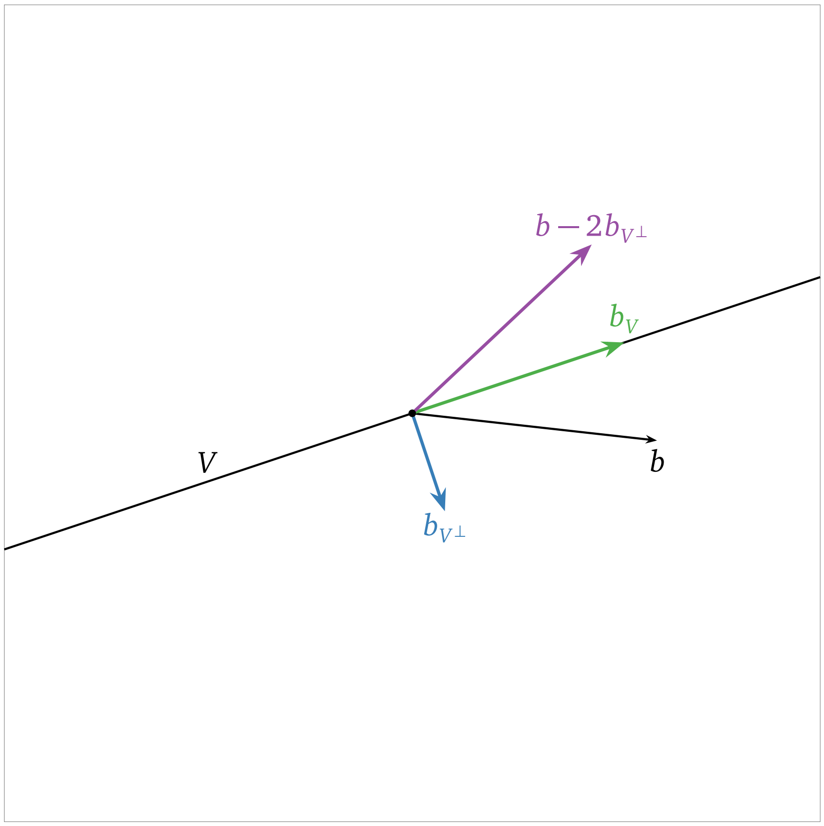
[12 points]

A line V and a vector b are drawn below. Draw *and label*:

a) The orthogonal projection b_V.

b) The projection onto the orthogonal complement $b_{V^\perp}$.

c) The vector $b - 2b_{V^\perp}$.



Problem 5.

[15 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 2 \\ 0 \\ -2 \\ 0 \end{pmatrix} \qquad v_4 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the subspace $W = \text{Span}\{v_1, v_2, v_3, v_4\}$.

[**Hint:** in this problem it is helpful, but not necessary, to use the fact that $\{v_1, v_2, v_3\}$ is orthogonal.]

a) Find a linear relation among v_1, v_2, v_3, v_4.

b) What is the dimension of W?

c) List all subsets of $\{v_1, v_2, v_3, v_4\}$ that form a basis for W.

Solution.

a)

$$v_1 + v_2 + v_3 - v_4 = 0$$

b)

$$\dim(W) = 3$$

c) $\{v_1, v_2, v_3\}$,$\{v_1, v_2, v_4\}$,$\{v_1, v_3, v_4\}$,$\{v_2, v_3, v_4\}$

Problem 6. [16 points]

All of the following statements are false. Find a counterexample.

a) If Q has orthonormal columns, then QQ^T is the identity matrix.

b) If V and W are subspaces of $\mathbf{R}^n$ and every vector in V is orthogonal to every vector in W, then $V = W^{\perp}$.

c) A matrix with orthogonal columns has full row rank.

d) If A is an $m \times n$ matrix and $A^T A$ is invertible, then A has rank m.

e) If Q has orthogonal columns, then $\|Qx\| = \|x\|$ for any vector x.

Solution.

a) Any non-square matrix with orthonormal columns is a counterexample.

b) For example, $V = \text{Span}\{(1,0,0)\}$ and $W = \text{Span}\{(0,1,0)\}$.

c) Any non-square matrix with orthogonal columns is a counterexample.

d) Any non-square matrix with full column rank is a counterexample.

e) Choose a matrix Q with orthogonal columns whose first column does not have length 1; then $\|Qe_1\| \neq 1 = \|e_1\|$.