MATH 218D-1
PRACTICE MIDTERM EXAMINATION 3

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [20 points]

a) Verify that the symmetric matrix

$$S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

is positive-definite without finding its eigenvalues.

b) Compute the characteristic polynomial of the matrix in part **a)**. (Do not factor it.)

c) Consider the symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & 4 \\ -4 & 4 & 5 \end{pmatrix}.$$

Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$. The eigenvalues of S are $9, 3$, and -3.

Solution.

a) This can be accomplished by finding the LU decomposition:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{pmatrix}.$$

b)

$$p(\lambda) = -\lambda^3 + 12^2 - 33\lambda + 6$$

c)

$$Q = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \qquad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Problem 2.

[20 points]

Consider the difference equation

$$\begin{aligned} x_{n+1} &= 2x_n - y_n & \qquad x_0 &= 1 \\ y_{n+1} &= \tfrac{3}{2}x_n - \tfrac{1}{2}y_n & \qquad y_0 &= 2. \end{aligned}$$

a) Find a matrix A such that

$$A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.$$

b) Find the eigenvalues of A, and find corresponding eigenvectors.

c) Find a formula for $\binom{x_n}{y_n}$ in terms of n.

d) What is $\lim_{n\to\infty} \binom{x_n}{y_n}$?

Solution.

a) The matrix is $A = \begin{pmatrix} 2 & -1 \\ 3/2 & -1/2 \end{pmatrix}$.

b) The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and corresponding eigenvectors are $w_1 = \binom{1}{1}$ and $w_2 = \binom{2}{3}$.

c) We have $\binom{x_0}{y_0} = -w_1 + w_2$, so

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -A^n w_1 + A^n w_2 = -w_1 + \frac{1}{2^n} w_2 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2^n}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

d) The limit is $-w_1 = -\binom{1}{1}$.

Problem 3. [10 points]

Solve the following initial value problem:

$$\begin{aligned} u_1' &= 2u_1 - u_2 & \qquad u_1(0) &= 1 \\ u_2' &= \tfrac{3}{2}u_1 - \tfrac{1}{2}u_2 & \qquad u_2(0) &= 2. \end{aligned}$$

Solution.

$$\begin{aligned} u_1(t) &= -e^t + 2e^{t/2} \\ u_2(t) &= -e^t + 3e^{t/2}. \end{aligned}$$

Problem 4.

[20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A symmetric matrix satisfying

$$S\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\4\\6\end{pmatrix} \quad \text{and} \quad S\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}.$$

b) A 2×2 matrix whose 1-eigenspace is the line $x+2y=0$ and whose 2-eigenspace is the line $x+3y=0$.

c) A 2×2 matrix that is neither invertible nor diagonalizable.

d) A 2×2 non-invertible matrix with eigenvalue $2+3i$.

e) A 2×2 matrix A that is diagonalizable over $\mathbf{R}$, such that A^2 is not diagonalizable.

Solution.

a) Does not exist: eigenvectors with different eigenvalues would have to be orthogonal.

b) This matrix satisfies

$$A=\begin{pmatrix}-2&-3\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}-2&-3\\1&1\end{pmatrix}^{-1}=\begin{pmatrix}4&6\\-1&-1\end{pmatrix}.$$

c) One example is $\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

d) Does not exist: the other eigenvalue would be $2-3i$, so 0 is not an eigenvalue.

e) Does not exist: if $A=CDC^{-1}$ then $A^2=CD^2C^{-1}$.

Problem 5. [5 points]

Let A be an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2.$$

Which of the following can you determine from this information? (Select all that apply.)

(1) The number n.
(2) The trace of A.
(3) The determinant of A.
(4) The rank of A.
(5) Whether A is symmetric.
(6) Whether A is diagonalizable.
(7) The eigenvalues of A.

Solution.

You can determine (1), (2), (3), (4), and (7).

(1) $n = \deg(p) = 4$.
(2) $\mathrm{Tr}(A)$ is the sum of the eigenvalues (with multiplicity), which is $2 + 3 + 3 = 8$.
(3) $\det(A)$ is the product of the eigenvalues (with multiplicity), which is 0.
(4) The null space is the 0-eigenspace, which has algebraic multiplicity 1, hence also geometric multiplicity 1. Therefore $\dim \mathrm{Nul}(A) = 1$, so $\mathrm{rank}(A) = 4 - 1 = 3$.
(5) You can't tell. Both of these matrices have characteristic polynomial $p(\lambda)$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

(6) You can't tell. The first matrix above is diagonal, and the second is not diagonalizable.
(7) The eigenvalues are $0, 2$, and 3.

Problem 6.

[10 points]

A certain diagonalizable 2×2 matrix A has eigenvalues 1 and 2, with eigenspaces drawn below.

a) Draw Ax and Ay on the diagram.

b) Draw the vector $w = \lim_{n\to\infty} A^n x / \|A^n x\|$: that is, eventually $A^n x$ points in the direction of the unit vector w. (Let's say that 1cm on your paper is one unit.)