MATH 218D-1
MIDTERM EXAMINATION 3

Name		**Duke Email**	@duke.edu

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

Problem 1. [20 points]

Compute the determinants of the following matrices.

a) $\det\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = 0$

b) $\det\begin{pmatrix} 5 & 0 & 0 \\ -3 & 0 & 0 \\ 8 & 5 & -1 \end{pmatrix} = 0$

c) $\det\begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 7 & 1 \end{pmatrix} = -2 \cdot 2 \cdot 5 \cdot 7$

d) $\det\begin{pmatrix} 1 & -2 & 3 \\ 2 & 0 & -6 \\ 1 & 0 & -3 \end{pmatrix} = 0$

Problem 2.

[20 points]

Consider the matrix

$$A = \frac{1}{10}\begin{pmatrix} 11 & -3 \\ -3 & 19 \end{pmatrix}.$$

a) Find the eigenvalues of A and *orthonormal* eigenvectors.

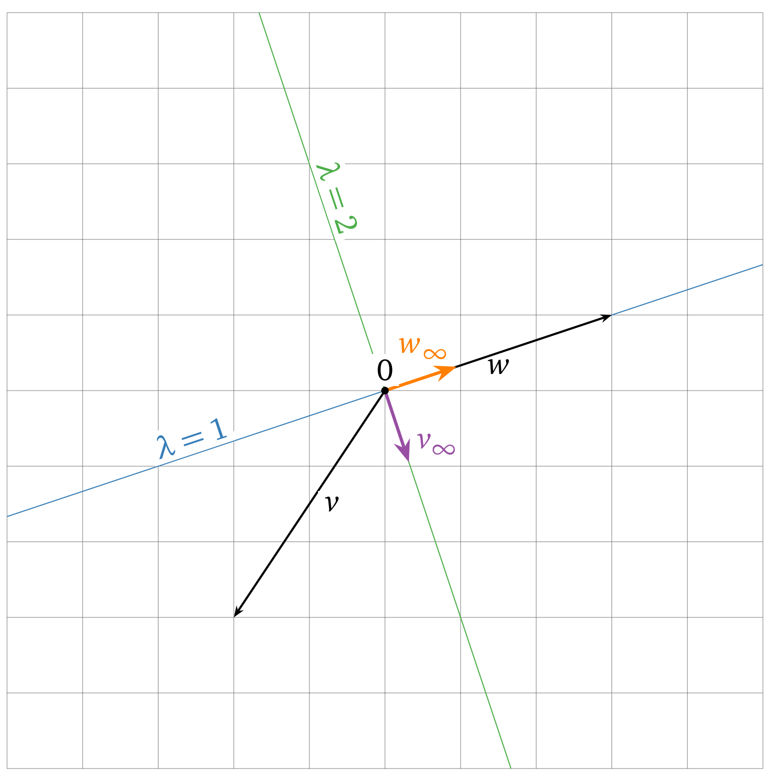
$$\lambda_1 = 1 \qquad w_1 = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 2 \qquad w_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

b) Draw the eigenspaces of A in the grid below, and label them with the corresponding eigenvalues. Be precise!

c) Vectors v and w are shown in the picture. Draw and label the vectors

$$v_\infty = \lim_{k\to\infty} \frac{A^k v}{\|A^k v\|} \qquad \text{and} \qquad w_\infty = \lim_{k\to\infty} \frac{A^k w}{\|A^k w\|}.$$



Problem 3.

[20 points]

In this problem, you need not explain your answers; just write them **in the spaces provided**.

Let A be an $n \times n$ matrix.

a) Which **one** of the following statements is correct? [2]

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct? [2]

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

c) Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (List all that apply.) [1, 2, 3]

1. A matrix with three distinct real eigenvalues.
2. A symmetric matrix with two real eigenvalues.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

d) Give an example of a 2×2 matrix that is neither invertible nor diagonalizable.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Problem 4.

[20 points]

a) Compute the characteristic polynomial of the following matrix. Do not factor it!

$$\begin{pmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & 2 & 4 \end{pmatrix} \rightsquigarrow \quad p(\lambda) = -\lambda^3 + 9\lambda^2 - 24\lambda + 18$$

b) Consider the matrix

$$A = \begin{pmatrix} 3 & 7 & -11 \\ 6 & 22 & -33 \\ 4 & 14 & -21 \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. Find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

$$C = \begin{pmatrix} -7/2 & 11/2 & 1/2 \\ 1 & 0 & 3/2 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

c) In part **b)**, is it possible to find an *orthogonal* matrix Q and a diagonal matrix D such that $A = QDQ^T$? Why or why not?

No: any matrix of the form QDQ^T is symmetric.

Problem 5.

[20 points]

Consider the following initial value problem:

$$\begin{aligned} u_1' &= 2u_1 - u_2 \qquad & u_1(0) = 2 \\ u_2' &= 3u_1 - u_2 \qquad & u_2(0) = 3. \end{aligned}$$

a) Let $u(t) = (u_1(t), u_2(t))$. Find the matrix A such that $u' = Au$.

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix}$$

b) Find the eigenvalues of A.

$$\lambda = \frac{1}{2}(1 + i\sqrt{3}) \qquad \overline{\lambda} = \frac{1}{2}(1 - i\sqrt{3})$$

c) For each eigenvalue λ_i, find the corresponding eigenvector w_i whose first coordinate is 1.

$$w = \begin{pmatrix} 1 \\ \frac{1}{2}(3 - i\sqrt{3}) \end{pmatrix} \qquad \overline{w} = \begin{pmatrix} 1 \\ \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix}$$

d) Express $u(0) = (2, 3)$ as a linear combination of the eigenvectors you found in **c)**.

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = w + \overline{w}.$$

e) Solve the initial value problem $u' = Au$, $u(0) = (2, 3)$. Your answer should involve only real numbers.

$$u_1 = 2e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)$$

$$u_2 = e^{t/2} \left(3\cos\left(\frac{\sqrt{3}}{2} t\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2} t\right)\right).$$