Homework #1

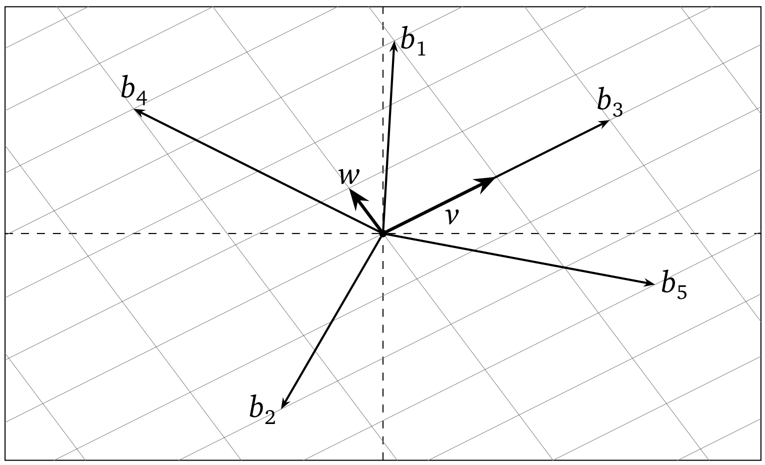
due Monday, August 30, at 11:59pm

1. Consider the vectors

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw the 16 linear combinations $cv + dw$ $(c, d = -1, 0, 1, 2)$ in the xy-plane.

2. Certain vectors v, w in $\mathbf{R}^2$ are drawn below. Express each of b_1, b_2, b_3, b_4, b_5 as a linear combination of v, w.



3. If

$$v + w = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad \text{and} \quad v - w = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

compute and draw the vectors v and w.

4. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u + v + w$ and $u + 2v - w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Explain why every linear combination of u, v, w is also a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to _____?

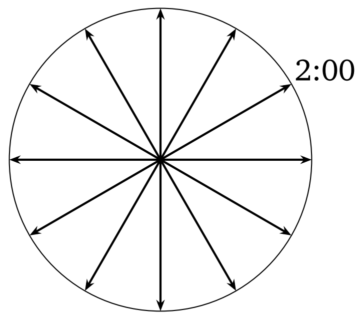
e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

5. Consider the vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Draw a picture of all of the linear combinations $au + bv$ for real numbers a, b satisfying $0 \leq a \leq 1$ and $0 \leq b \leq 1$.

6. Consider the vectors pointing towards the numbers on a clock:



a) What is the sum of all twelve of these vectors?

b) If the 2:00 vector is removed, why do the remaining vectors add to 8:00?

7. Find two *different* triples (x, y, z) such that

$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

How many such triples are there?

8. Decide if each statement is true or false, and explain why.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c) If v, w are two vectors in $\mathbf{R}^2$, then any other vector b in $\mathbf{R}^2$ is a linear combination of v and w.

9. Consider the following vectors:

$$u = \begin{pmatrix} -.6 \\ .8 \end{pmatrix} \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Compute the lengths $\|u\|, \|v\|$, and $\|w\|$.

b) Compute the lengths $\|2u\|, \|-v\|$, and $\|3w\|$.

c) Find the unit vectors in the directions of u, v, and w.

d) Compute the dot products $u \cdot v, u \cdot w$, and $v \cdot w$. Verify that they are the same as $v \cdot u, w \cdot u$, and $w \cdot v$, respectively.

e) Check the Schwartz inequalities $|u \cdot v| \leq \|u\| \, \|v\|$ and $|v \cdot w| \leq \|v\| \, \|w\|$.

f) Find the angles between u and v and between v and w.

g) Find the distance from v to w.

h) Find unit vectors u', v', w' orthogonal to u, v, w, respectively.

10. Suppose that v and w are *unit* vectors. Compute the following dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v+w) \cdot (v-w)$ **c)** $(v+2w) \cdot (v-2w)$.

11. Decide if each statement is true or false, and explain why.

a) If $u = (1,1,1)$ is orthogonal to v and to w, then v is parallel to w.

b) If u is orthogonal to $v+w$ and to $v-w$, then u is orthogonal to v and w.

c) If u and v are orthogonal unit vectors then $\|u-v\| = \sqrt{2}$.

d) If $\|u\|^2 + \|v\|^2 = \|u+v\|^2$, then u and v are orthogonal.

12. Find nonzero vectors v and w that are orthogonal to $(1,1,1)$ and to each other.

13. What is the length of the vector $v = (1,1,\ldots,1)$ in n dimensions?

14. If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and largest possible values of $\|v-w\|$? What are the smallest and largest possible values of $v \cdot w$? Justify your answer using the algebra of dot products.

15. **a)** If $v \cdot w < 0$, what does that say about the angle between v and w?

b) Find three vectors u, v, w in the xy-plane such that $u \cdot v < 0$, $u \cdot w < 0$, and $v \cdot w < 0$.

16. Compute the following matrix-vector products using *both* the row-first and column-first methods. If the product is not defined, explain why.

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 0 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 7 & 2 & 4 \\ 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 4 \\ -2 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 6 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 & 6 & -1 \end{pmatrix}$$

17. Suppose that $u = (x, y, z)$ and $v = (a, b, c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

18. Consider the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.$$

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$ **b)** $B - 3A$ **c)** AC **d)** B^2

e) $A + 2B$ **f)** $C - E$ **g)** EB **h)** D^2

19. Compute the product

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

in three ways:

a) Using the "column first" method.

b) Using the "rows first" method.

c) Using the outer product form.

20. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & h \end{pmatrix}.$$

What value(s) of h, if any, will make $AB = BA$?

21. Consider the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & -8 \\ 5 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Verify that $AC = BC$ and yet $A \neq B$.

22. For the following matrices A and B, compute $AB, A^T, B^T, B^T A^T$, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute $A^T B^T$?

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.$$

23. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries.

System of Equations	**Matrix Equation**	**Augmented Matrix**
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$		
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	
		$\left(\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 3 & -1 & -2 \\ 1 & -3 & -4 & -3 \\ 6 & 5 & -1 & -8 \end{array}\middle\vert\begin{array}{c} 2 \\ 4 \\ 2 \\ 1 \end{array}\right)$

24. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Use row operations to eliminate x_1 from all but the first equation.

b) Use row operations to modify the system so that x_2 only appears in the first and second equations (and x_1 still only appears in the first).

c) Solve for x_3, then for x_2, then for x_1. What is the solution?

25. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$