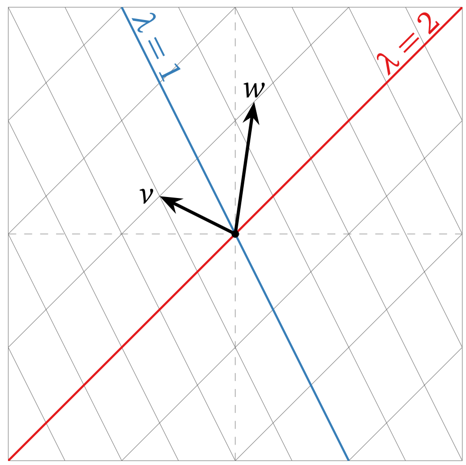
Homework #11

due Monday, November 8, at 11:59pm

1. A certain 2×2 matrix A has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.

a) Draw Av, $A^2 v$, and Aw.

b) Compute the limit of $A^n v / \|A^n v\|$ as $n \to \infty$.



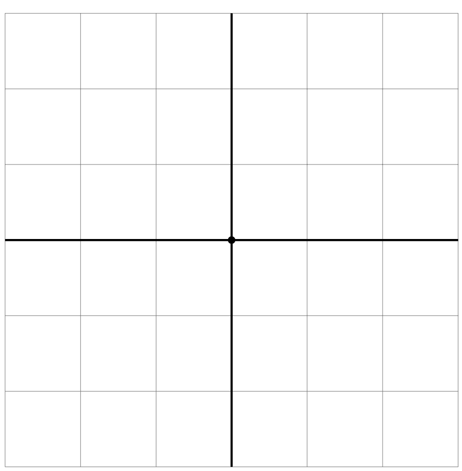
2. A certain diagonalizable 2×2 matrix A is equal to CDC^{-1}, where C has columns w_1, w_2 pictured below, and $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$.

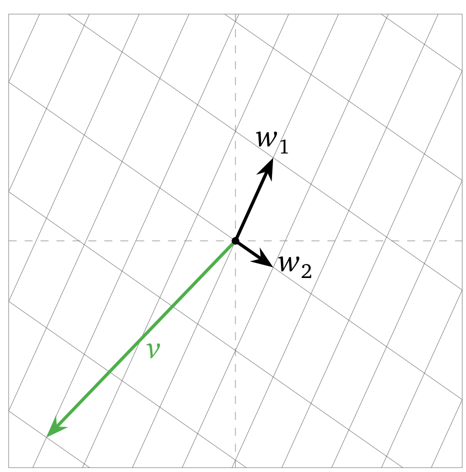
a) Draw $C^{-1}v$ on the left.

b) Draw $DC^{-1}v$ on the left.

c) Draw $Av = CDC^{-1}v$ on the right.

d) What happens to $A^n v$ as $n \to \infty$?





3. Compute the following complex numbers.

a) $(1+i)+(2-i)$ **b)** $(1+i)(2-i)$ **c)** $\overline{2-i}$ **d)** $\dfrac{1+i}{2-i}$

e) $|1+i|$ **f)** $2e^{2\pi i/3}$ **g)** $5e^{3\pi i}$

4. Express each complex number in polar coordinates $re^{i\theta}$.

a) $1+i$ **b)** $\dfrac{-1+i\sqrt{3}}{2}$ **c)** $-\sqrt{3}-3i$ **d)** $\dfrac{1}{1+i}$ **e)** $\left(1-i\sqrt{3}\right)^n$

5. For which numbers θ is $e^{i\theta}=1$? What about -1?

6. For each matrix A and each vector x, decide if x is an eigenvector of A, and if so, find the eigenvalue λ.

a) $\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix}$ **b)** $\begin{pmatrix} -4 & 13 & 13 \\ 2 & -2 & -4 \\ -4 & 8 & 10 \end{pmatrix}, \begin{pmatrix} 1+5i \\ -2i \\ 4i \end{pmatrix}$

c) $\begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ -2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2+i \\ 1 \\ -i \end{pmatrix}$

Careful! It is difficult to recognize by inspection if two complex vectors are (complex) scalar multiples of each other.

7. For each 2×2 matrix A, **i)** compute the characteristic polynomial, **ii)** find all (real and complex) eigenvalues, and **iii)** find a basis for each eigenspace, using Problem 3 on Homework 10 when applicable. **iv)** Is the matrix diagonalizable (over the complex numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A=CDC^{-1}$.

a) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 5 \\ -10 & 7 \end{pmatrix}$

8. Diagonalize the following matrix over the complex numbers:

$$A=\begin{pmatrix} 1 & 4 & -6 \\ -6 & 7 & -22 \\ -2 & 1 & -5 \end{pmatrix}.$$

One eigenvalue is $\lambda=-1$.

9. A certain forest contains a population of rabbits and a population of foxes. If there are r_n rabbits and f_n foxes in year n, then

$$\begin{aligned} r_{n+1} &= 3r_n - f_n \\ f_{n+1} &= r_n + 2f_n: \end{aligned}$$

in other words, each rabbit produces three baby rabbits on average, but there is some loss due to predation by foxes; each fox produces two babies on average, but this is increased with ample prey.

a) Let $v_n = \binom{r_n}{f_n}$. Find a matrix A such that $v_{n+1} = Av_n$.

b) Find an eigenbasis of A. (The eigenvectors and eigenvalues will be complex.)
[**Hint:** Part **d)** will be easier if you choose the eigenvectors with first coordinate equal to 1.]

c) Suppose that $r_0 = 2$ and $f_0 = 1$. Find closed formulas for r_n and f_n. Find a formula for r_n involving only real numbers. (This latter formula can involve an arctan.)

d) In this model, the populations do not stabilize. How many years will it take for the foxes to eat all of the rabbits?

In general, any 2×2 difference equation with a complex eigenvalue will exhibit oscillation centered at zero. This phenomenon can be described explicitly, but is beyond the scope of this course.

10. **a)** Let A be an $n \times n$ matrix. Prove that λ is an eigenvalue of A with geometric multiplicity n if and only if $A = \lambda I_n$.

b) Find a non-diagonal 2×2 matrix such that 1 is an eigenvalue with algebraic multiplicity 2.

11. Find examples of real 2×2 matrices A with the following properties.

a) A is invertible and diagonalizable over the real numbers.

b) A is invertible but not diagonalizable over the complex numbers.

c) A is diagonalizable over the real numbers but not invertible.

d) A is neither invertible nor diagonalizable over the complex numbers.

This shows that *invertibility and diagonalizability have nothing to do with each other*.

12. Let A be an $n \times n$ matrix.

a) Show that the product of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\det(A)$.

b) [**Optional**] Show that the sum of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to $\mathrm{Tr}(A)$.

(Both of these are identities involving the characteristic polynomial of A.)

13. For each matrix in Problem 5(a)–(c) on Homework 10, compute the algebraic and geometric multiplicity of each eigenvalue. What does your answer say about diagonalizability? **Optional:** do (d)–(g) as well.

14. Give an example of each of the following, or explain why no such example exists. All matrices should have real entries.

a) A 3×3 matrix with eigenvalues $0, 1, 2$, and corresponding eigenvectors

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\-1\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\1\end{pmatrix}.$$

b) A 4×4 matrix having eigenvalue 2 with algebraic multiplicity 2 and geometric multiplicity 3.

c) A 3×3 matrix with one complex eigenvalue and two real eigenvalues.

d) A 2×2 matrix A such that A^2 is diagonalizable over the real numbers but A is not diagonalizable, even over the complex numbers.

[**Hint:** try a nonzero matrix A such that $A^2 = 0$.]

15. Decide if each statement is true or false, and explain why.

a) If A and B are diagonalizable $n \times n$ matrices, then so is AB.

b) An $n \times n$ matrix with n (different) eigenvalues is diagonalizable.

c) An $n \times n$ matrix is diagonalizable if it has n eigenvalues, counted with algebraic multiplicity.

d) Any 2×2 real matrix with a complex eigenvalue is diagonalizable over the complex numbers.

e) Any 3×3 real matrix with a complex eigenvalue is diagonalizable over the complex numbers.

f) Any 4×4 real matrix with a complex eigenvalue is diagonalizable over the complex numbers.

g) Any 2×2 real matrix has a real eigenvalue.

h) Any 3×3 real matrix has a real eigenvalue.

i) Any $n \times n$ matrix has a (real or complex) eigenvalue.

j) If the characteristic polynomial of A is $-(\lambda^3 - 1) = -(\lambda^2 + \lambda + 1)(\lambda - 1)$, then the 1-eigenspace of A is a line.