Homework #12

due Monday, November 15, at 11:59pm

1. Solve the following initial value problems. Your solutions should involve only real numbers.

a) $\begin{cases} u_1' = u_1 - 2u_2 \\ u_2' = u_1 + 4u_2 \end{cases}$ $\quad u_1(0) = -3, \quad u_2(0) = 2$

b) $\begin{cases} u_1' = 3u_1 - u_2 \\ u_2' = u_1 + 2u_2 \end{cases}$ $\quad u_1(0) = 4, \quad u_2(0) = 2$

2. Solve the following initial value problem.

$$p''(t) = -2p(t) + 3p'(t) \qquad p(0) = 1 \qquad p'(0) = -1.$$

3. Use the matrix exponential to solve the following initial value problem.

$$\begin{cases} u_1' = 2u_2 + u_3 & u_1(0) = 2 \\ u_2' = -u_3 & u_2(0) = 3 \\ u_3' = 0 & u_3(0) = -1. \end{cases}$$

(This is one of the few instances where the matrix exponential leads to a computable solution!)

4. For each symmetric matrix S, find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

a) $\begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$ **c)** $\begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}$

d) $\begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$

The eigenvalues in **d)** are $3, 6, 9$ and in **e)** are $-9, 9$.

5. For each matrix S of Problem 4, decide if S is positive-semidefinite, and if so, compute its positive-semidefinite square root $\sqrt{S} = Q\sqrt{D}Q^T$. Verify that $(\sqrt{S})^2 = S$.

Remark: Since $\sqrt{S}$ is also symmetric, we have $S = \sqrt{S}^T \sqrt{S}$, so this is another way to factorize a positive-semidefinite matrix as $A^T A$.

6. Consider the matrix

$$S = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

of Problem 4(d). Write S in the form $\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T$ for numbers $\lambda_1, \lambda_2, \lambda_3$ and orthonormal vectors q_1, q_2, q_3.

[**Hint:** Use the columns of Q. Why does this work?]

7. Find *all possible* orthogonal diagonalizations

$$\frac{1}{5}\begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} = QDQ^T.$$

8. Let S be a symmetric matrix such that $S^k = 0$ for some $k > 0$. Show that $S = 0$. [**Hint:** Use Problem 17 on Homework 10.]

9. Let S be a symmetric orthogonal 2×2 matrix.

a) Show that $S = \pm I_2$ if it has only one eigenvalue.

b) Suppose that S has two eigenvalues. Show that S is the matrix for the reflection over a line L in $\mathbf{R}^2$. (Recall that the reflection over a line L is given by $R_L = I_2 - 2P_{L^\perp}$.)
[**Hint:** Write S as $\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$, and use the projection formula to write I_2 and $P_{L^\perp}$ in this form as well.]

10. **a)** Let S be a diagonalizable (over $\mathbf{R}$) $n \times n$ matrix with orthogonal eigenspaces: that is, eigenspaces with different eigenvalues are orthogonal subspaces. Prove that S is symmetric.

[**Hint:** choose *orthonormal* bases for each eigenspace.]

b) Let S be a matrix that can be written in the form

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

for some vectors $q_1, q_2, \ldots, q_n$. Prove that S is symmetric.

c) Let V be a subspace of $\mathbf{R}^n$, and let P_V be the projection matrix onto V. Use **a)** or **b)** to prove that P_V is symmetric. (We proved this in class in Week 6 using the formula $P_V = A(A^TA)^{-1}A^T$.)

11. For which matrices A is $S = A^TA$ positive-definite? If S is not positive-definite, find a vector x such that $x^TSx = 0$. In any case, do not compute S!

a) $\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

12. **a)** If S is positive-definite and C is invertible, show that CSC^T is positive-definite.

b) If S and T are positive-definite, show that $S + T$ is positive-definite.

c) If S is positive-definite, show that S is invertible and that S^{-1} is positive-definite.

[**Hint:** For **a)** and **b)** use the positive-energy characterization of positive-definiteness; for **c)** use the positive-eigenvalue characterization.]

13. Let S be a positive-definite matrix.

a) Show that the diagonal entries of S are positive.

[**Hint:** compute $e_i^T S e_i$.]

b) Show that the diagonal entries of S are all greater than or equal to the smallest eigenvalue of S.

[**Hint:** if not, apply **a)** to $S - aI_n$ for a diagonal entry a that is smaller than all eigenvalues.]

14. Decide if each statement is true or false, and explain why. All matrices are real.

a) A symmetric matrix is diagonalizable.

b) If A is any matrix then $A^T A$ is positive-semidefinite.

c) A symmetric matrix with positive determinant is positive-definite.

d) If $A = CDC^{-1}$ for a diagonal matrix D and a non-orthogonal invertible matrix C, then A is not symmetric.

e) A positive-definite matrix has the form $A^T A$ for a matrix A with full column rank.

f) The only positive-definite projection matrix is the identity.

g) All eigenvalues of a positive-definite symmetric matrix are positive real numbers.