

## Homework #2

due Monday, September 6, at 11:59pm

1. Which of the following matrices are not in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

2. Describe all possible nonzero  $2 \times 2$  matrices in RREF.
3. Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use [Rabinoff's Reliable Row Reducer](#).

$$\text{a) } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{b) } \left( \begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array} \right) \quad \text{c) } \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$
$$\text{d) } \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \quad \text{e) } \left( \begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array} \right) \quad \text{f) } \left( \begin{array}{cccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right)$$

4. Determine *how many* solutions each system of equations has. (Do not find the solutions.) [Hint: use Problem 3.]

$$\text{a) } \begin{cases} x_1 + x_2 = 1 \\ x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 2 \end{cases} \quad \text{b) } \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 9x_3 = 1 \end{cases}$$
$$\text{c) } \begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

5. Use Gaussian elimination and back-substitution to solve

$$\text{a) } \begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ x_2 + 2x_3 = 3 \end{cases} \quad \text{b) } \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 8x_3 = 1. \end{cases}$$

What happens if you replace 8 by 9 in (b)?

6. Three planes can fail to have an intersection point, even if no planes are parallel. Consider the two planes  $A : x + y + z = 0$  and  $B : x - 2y - z = 1$ . Use the tool here:

<https://technology.cpm.org/general/3dgraph/>

to visualize these two planes, then answer the following questions:

- What is the shape of the intersection  $A \cap B$  of the two?
- Use the equations of  $A$  and  $B$  to construct a third plane  $C$  whose intersection with the two is exactly the same as  $A \cap B$ . That is,  $A \cap B \cap C = A \cap B$  [**Hint:** how can you create a system of three equations with fewer than three pivots?]
- Find a fourth plane  $D$  such that  $A \cap D$ , and  $B \cap D$  are both non-empty, but  $A \cap B \cap D$  is empty. That is,  $D$  should intersect both  $A$  and  $B$ , but the three should never meet. [**Hint:** make the system inconsistent!]

For both the last two parts, I strongly suggest you use the tool linked above to draw the planes and see your answers!

7. The parabola  $y = ax^2 + bx + c$  passes through the points  $(1, 4)$ ,  $(2, 9)$ ,  $(-1, 6)$ . Find the coefficients  $a, b, c$ .
8. Find values of  $a$  and  $b$  such that the following system is **a)** inconsistent and **b)** consistent.

$$\begin{aligned}2x + ay &= 4 \\ x - y &= b\end{aligned}$$

9. Let  $A$  be a matrix in REF. Suppose that  $A$  has a pivot position in every row. Explain why the linear system  $Ax = b$  is consistent. [**Hint:** What happens when you do back-substitution?]
10. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:
- $R_2 += 2R_1$
  - $R_1 -= \frac{1}{2}R_3$
  - $R_3 \times = 2$
  - $R_3 \div = 2$
  - $R_1 \longleftrightarrow R_3$
  - $R_1 \longleftrightarrow R_2$

11. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

$$\text{a)} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{b)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{c)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$\text{d)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{e)} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

12. For each elementary matrix in Problem 10, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row op}} R_2 += R_1 \xrightarrow{\text{undo}} R_2 -= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

13. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

- a) Explain how to reduce  $A$  to a matrix  $U$  in REF using three row replacements.  
b) Let  $E_1, E_2, E_3$  be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the  $E_i$ :

$$U = \underline{\hspace{2cm}} A.$$

- c) Fill in the blank with a product involving the  $E_i^{-1}$ :

$$A = \underline{\hspace{2cm}} U$$

- d) Evaluate that product to produce a lower-triangular matrix  $L$  with ones on the diagonal such that  $A = LU$ .  
e) Explain how to reduce  $U$  to the  $3 \times 3$  identity matrix using three more row operations  $E_4, E_5, E_6$ .

- f) Fill in the blank with a product involving the  $E_i$ :

$$A^{-1} = \underline{\hspace{2cm}}.$$

14. Use the formula for the  $2 \times 2$  inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

$$\text{a)} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{b)} \begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \quad \text{c)} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

15. Compute the inverse of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why.

$$\text{a) } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix} \quad \text{c) } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

16. Consider the linear system

$$\begin{aligned} x_1 + x_2 &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 15 to solve for  $x_1, x_2, x_3$  in terms of  $b_1, b_2, b_3$ .

17. Decide if each statement is true or false, and explain why.

- If  $A$  and  $B$  are invertible  $n \times n$  matrices, then  $AB$  is invertible, and  $(AB)^{-1} = A^{-1}B^{-1}$ .
- An  $n \times n$  matrix with  $n$  pivots is invertible.
- An invertible  $n \times n$  matrix has  $n$  pivots.

18. Suppose that

$$A \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad A \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad A \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

What is  $A^{-1}$ ?

19. Consider the matrix

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}.$$

- Perform Gaussian elimination on  $A$  without using any row swaps. Write the REF matrix  $U$  you obtained.
- Write the elementary matrices  $E_1, E_2, E_3$  for the row operations you did in (a), with  $E_1$  corresponding to the first row operation.
- Compute the matrix  $L = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}$ . [Hint: Don't do elimination again! Recall that left-multiplication by  $E_i^{-1}$  "un-does" the  $i$ th row operation.]
- Verify that  $L$  is lower-unitriangular and that  $A = LU$ .