Homework #6

due Wednesday, October 6, at 11:59pm

1. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\mathrm{Col}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **b)** $\mathrm{Nul}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$ **c)** $\mathrm{Row}\begin{pmatrix}1&4&7\\2&5&8\\3&6&9\end{pmatrix}$

d) $\mathrm{Nul}\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ **e)** $\mathrm{Span}\left\{\begin{pmatrix}3\\1\\3\end{pmatrix},\begin{pmatrix}0\\-2\\1\end{pmatrix}\right\}$ **f)** $\mathrm{Col}\begin{pmatrix}1&2&0\\2&1&0\end{pmatrix}$

[**Hint:** solving **a)**–**d)** requires only one Gauss-Jordan elimination, and **f)** doesn't require any work.]

2. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\{(x,y,x)\colon x,y\in\mathbf{R}\}$.

b) $\{(x,y,z)\in\mathbf{R}^3\colon x=2y+z\}$.

c) The solution set of the system of equations $\begin{cases}x+\ \ y+z=0\\ x-2y-z=0.\end{cases}$

d) $\{x\in\mathbf{R}^3\colon Ax=2x\}$, where $A=\begin{pmatrix}0&6&8\\\frac12&0&0\\0&\frac12&0\end{pmatrix}$.

e) The subspace of all vectors in $\mathbf{R}^3$ whose coordinates sum to zero.

f) The intersection of the plane $x-2y-z=0$ with the xy-plane.

g) The line $\{(t,-t,t)\colon t\in\mathbf{R}\}$.

[**Hint:** Compare Problem 9 on Homework 5.]

3. For each pair of vectors v and w, draw $\mathrm{Span}\{v\}$, and compute and draw the projection p of w onto $\mathrm{Span}\{v\}$.

a) $v=\begin{pmatrix}1\\0\end{pmatrix}$, $w=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$ **b)** $v=\begin{pmatrix}-1\\1\end{pmatrix}$, $w=\begin{pmatrix}1\\1\end{pmatrix}$

(Here θ is any number, so w is just a vector on the unit circle.)

4. For each subspace V and vector b, compute the orthogonal projection b_V of b onto V by solving a normal equation $A^TAx=A^Tb$, and find the distance from b to V.

a) $V=\mathrm{Col}\begin{pmatrix}1&1\\1&0\\0&2\end{pmatrix}$ $b=\begin{pmatrix}1\\4\\3\end{pmatrix}$

b) $V = \mathrm{Col}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$

c) $V = \mathrm{Col}\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$

5. For each subspace V, compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to V.

a) $V = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\}$ **b)** $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

c) $V = \mathbf{R}^3$ **d)** $V = \{0\}$

[**Hint:** Only part **a)** requires any work.]

6. Compute the orthogonal decomposition $(3, 1, 3) = b_V + b_{V^\perp}$ with respect to each subspace of V of Problem 1(a)–(e).

[**Hint:** Only parts **a)** and **c)** require any work, and even **c)** doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

7. Let A be an $m \times n$ matrix, and let $b \in \mathbf{R}^m$ be a vector. Suppose that $A^T b = 0$. Compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ with respect to $V = \mathrm{Col}(A)$.

8. **a)** Find an implicit equation for the plane

$$\mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right\}.$$

[**Hint:** use Problem 1(a).]

b) Find implicit equations for the line $\{(t, -t, t) : t \in \mathbf{R}\}$.

[**Hint:** use Problem 2(g).]

9. Construct a matrix A with each of the following properties, or explain why no such matrix exists.

a) The column space contains $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

b) The row space contains $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

c) $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent, and $A^T \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 0$.

d) A 2×2 matrix A with no zero entries such that every row of A is orthogonal to every column.

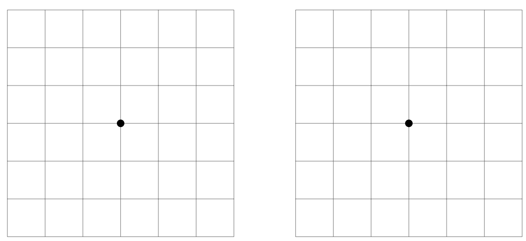
e) The sum of the columns of A is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and the sum of the rows of A is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

10. Suppose that S is a *symmetric matrix*: that is, a square matrix such that $S = S^T$. Explain why $\mathrm{Col}(S)$ is orthogonal to $\mathrm{Nul}(S)$.

11. Draw the four fundamental subspaces of the following matrices, in grids like below.

a) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

Row Picture *Column Picture*



12. The floor V and the wall W are not orthogonal subspaces, because they share a nonzero vector (along the line where they meet).

a) If $V = \mathrm{Col}(A)$ and $W = \mathrm{Col}(B)$ for

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 4 \\ 6 & 3 \\ 5 & 1 \end{pmatrix},$$

find a nonzero vector contained in both V and W.

[**Hint:** You want vectors x and y with $Ax = By$. Form the matrix $(A\ B)$.]

b) Generalize what you did in **a)** to explain why there do not exist orthogonal planes in $\mathbf{R}^3$.

13. Show that $A^T A = 0$ is only possible when $A = 0$.

14. Let Q be an $n \times n$ matrix such that $Q^T Q = I_n$ (so $Q^T = Q^{-1}$).

a) Show that the columns of Q are unit vectors.

b) Show that the columns of Q are orthogonal to each other.

c) Show that the *rows* of Q are also orthogonal unit vectors.

d) Find all 2×2 matrices Q such that $Q^T Q = I_2$.

Such a matrix Q is called *orthogonal*.

15. Construct a 3×3 matrix A, with no zero entries, whose columns are orthogonal to each other. Compute $A^T A$, and explain why this matrix is diagonal.

16. Explain why A has full column rank if and only if $A^T A$ is invertible.

17. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal.

b) If A is a 3×4 matrix, then $\mathrm{Col}(A)^\perp$ is a subspace of $\mathbf{R}^4$.

c) If A is any matrix, then $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$.

d) If A is any matrix, then $\mathrm{Row}(A) = \mathrm{Row}(A^T A)$.

e) If every vector in a subspace V is orthogonal to every vector in another subspace W, then $V = W^\perp$.

f) If x is in V and $V^\perp$, then $x = 0$.

g) If x is in a subspace V, then the orthogonal projection of x onto V is x.

h) If x is in the orthogonal complement of a subspace V, then the orthogonal projection of x onto V is x.