

Homework #9

due **Wednesday**, October 27, at 11:59pm

1. Compute the determinants of the following matrices using *Gaussian elimination*.

$$\text{a) } \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} \quad \text{b) } \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix}$$

$$\text{c) } \begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix} \quad \text{d) } \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

2. Suppose that

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10 \quad \text{and} \quad \det \begin{pmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{pmatrix} = 5.$$

Find the determinants of the following matrices.

$$\text{a) } \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \quad \text{b) } \begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix} \quad \text{c) } \begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix}$$

$$\text{d) } \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \quad \text{e) } \begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix} \quad \text{f) } \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$$\text{g) } 2 \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{h) } \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{i) } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$$

$$\text{j) } - \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{k) } \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^3 \quad \text{l) } \begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix}$$

$$\text{m) } \begin{pmatrix} a+2a' & b+2b' & c+2c' \\ d & e & f \\ g & h & i \end{pmatrix}$$

3. Compute

$$\det \left[\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} - \lambda I_3 \right]$$

where λ is an unknown real number. Your answer will be a function of λ .

4. Find $\det(E)$ when:
- E is the elementary matrix for a row replacement.
 - E is the elementary matrix for $R_i \times c$.
 - E is the elementary matrix for a row swap.

5. A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

What is $\det(A)$?

6.
 - Compute the determinants of the matrices in Problem 1 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!
 - Compute the determinants of the matrices in Problem 1(b) and (d) *again* using Sarrus' scheme.
 - For the matrix of Problem 1(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus' scheme. Did you get the determinant?

7. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

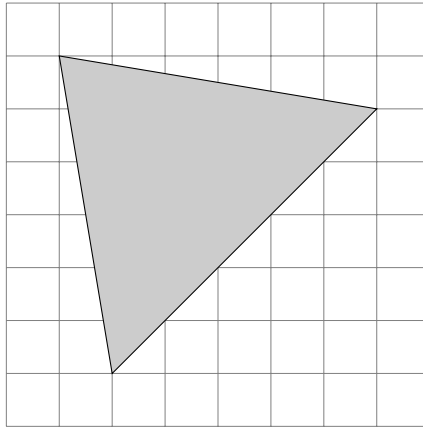
- Compute the cofactor matrix C of A .
 - Compute AC^T . What is the relationship between C^T and A^{-1} ?
8. Consider the $n \times n$ matrix F_n with 1's on the diagonal, 1's in the entries immediately below the diagonal, and -1 's in the entries immediately above the diagonal:

$$F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \dots$$

- Show that $\det(F_2) = 2$ and $\det(F_3) = 3$.
- Expand in cofactors to show that $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$.
- Compute $\det(F_4)$, $\det(F_5)$, $\det(F_6)$, $\det(F_7)$ using **b**).

This shows that $\det(F_n)$ is the n th *Fibonacci number*. (The sequence usually starts with 1, 1, 2, 3, ..., so our $\det(F_n)$ is the usual $n + 1$ st Fibonacci number.)

9. Let V be a subspace of \mathbf{R}^n and let P_V be the projection matrix onto V .
- Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.
 - Find $\det(P_V)$ when $V = \mathbf{R}^n$.
10. Let C be the *hypercube* in \mathbf{R}^4 with corners $(\pm 1, \pm 1, \pm 1, \pm 1)$. Compute the volume of C .
11. Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



12. Use a cross product to find an implicit equation for the plane

$$V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$

Compare Problem 8(a) in Homework 6.

13. a) Let $v = (a, b)$ and $w = (c, d)$ be vectors in the plane, and let $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. By taking the cross product of $(a, b, 0)$ and $(c, d, 0)$, explain how the right-hand rule determines the sign of $\det(A)$.
- b) Using the identity

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

explain how the right-hand rule determines the sign of a 3×3 determinant.

14. Decide if each statement is true or false, and explain why.
- $\det(A + B) = \det(A) + \det(B)$.
 - $\det(ABC^{-1}) = \frac{\det(A)\det(B)}{\det(C)}$.

- c) $\det(AB) = \det(BA)$.
- d) $\det(3A) = 3 \det(A)$.
- e) If A^5 is invertible then A is invertible.
- f) The determinant of A is the product of its diagonal entries.
- g) If the columns of A are linearly dependent, then $\det(A) = 0$.
- h) The determinant of the cofactor matrix of A equals the determinant of A .
- i) If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.
- j) $u \times v = v \times u$.
- k) If $u \times v = 0$ then $u \perp v$.