Eigenvalues & Eigenvectors

This is a core concept in linear algebra.
It's the tool used to study, among other things:

- Difference equations
- Differential equations
- Markov chains
- Stochastic processes

It will also be needed to understand the SVD.

Running Example

In a population of rabbits:

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- 1/4 survive their 1st year
- 1/2 survive their 2nd year
- Max lifespan is 3 years
- 1-year old rabbits have an average of 13 babies
- 2-year old rabbits have an average of 12 babies

Problem: Describe the long-term behavior of this system.

Let's give names to the state of the system in year n:

x_n = # babies in year n
y_n = # 1-year-olds in year n
z_n = # 2-year-olds in year n

$$v_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$$

The rules say:

$$x_{n+1} = 13y_n + 12z_n$$
$$y_{n+1} = \frac{1}{4}x_n$$
$$z_{n+1} = \frac{1}{2}y_n$$

As a matrix equation,

$$v_{n+1} = A v_n \qquad A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$$

This is an example of a difference equation; the A is the state change matrix. (It relates the state in year n to the state in year $n+1$.)

What happens in 100 years?

$$v_{100} = Av_{99} = A \cdot Av_{98} = \cdots = A^{100}v_0$$

$$\boxed{\text{Solving a difference equation means describing } A^k v_0 \text{ for large values of } k.}$$

Observation: If $v_0 = (32, 4, 1)$ then

$$Av_0 = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 64 \\ 8 \\ 2 \end{pmatrix} = 2v_0$$

So $A^2 v_0 = A(Av_0) = A(\lambda v_0) = \lambda A v_0 = \lambda^2 v_0$

$A^3 v_0 = A(A^2 v_0) = A(\lambda^2 v_0) = \lambda^2 A v_0 = \lambda^3 v_0$

$\vdots$

$A^k v_0 = \lambda^k v_0$

So if $Av_0 = (\text{scalar})\, v_0$ then it's easy to describe $A^k v_0 = (\text{scalar})^k v_0$ for any k!

Def: An *eigenvector* of a square matrix A is a *nonzero* vector v such that

$$Av = \lambda v \quad \text{for a scalar } \lambda.$$

The scalar λ is the associated *eigenvalue*.
We also say v is a *λ-eigenvector*

♫ eigenvector song ♫

$$\boxed{\text{If } Av = \lambda v \text{ then } A^k v = \lambda^k v \text{ for all } k}$$

Eg:
$$\begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 64 \\ 8 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}$$

So $(32, 4, 8)$ is an *eigenvector* with *eigenvalue* 2.

→ This means if you start with 32 babies, 4 1-year rabbits, and 1 2-year rabbit, then the population exactly doubles each year.

Next time: What if you *don't* start with an eigenvector?

Geometrically, an eigenvector of A is a nonzero vector v such that

$$\boxed{Av \text{ lies on the line thru the origin and } v.}$$

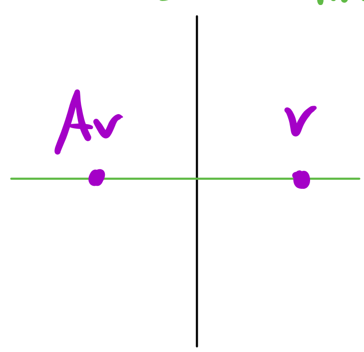
Eg: $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$: flip over y-axis.

Where are the eigenvectors?

- $v = \begin{pmatrix} x \\ 0 \end{pmatrix} \rightsquigarrow Av = \begin{pmatrix} -x \\ 0 \end{pmatrix} = -v$

  The (nonzero) vectors on the x-axis are eigenvectors with eigenvalue -1.

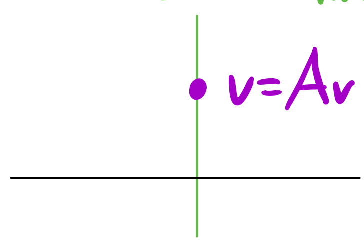
Av & v are on the same line.



- $v = \begin{pmatrix} 0 \\ y \end{pmatrix} \rightsquigarrow Av = \begin{pmatrix} 0 \\ y \end{pmatrix} = 1 \cdot v$

  The (nonzero) vectors on the y-axis are eigenvectors with eigenvalue $+1$.

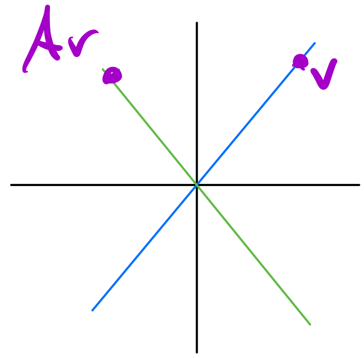
Av & v are on the same line.



- $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \neq 0$

  $\rightsquigarrow Av = \begin{pmatrix} -x \\ y \end{pmatrix}$ is not a multiple of v

Av & v are on different lines.



So we've found all eigenvectors (& eigenvalues).

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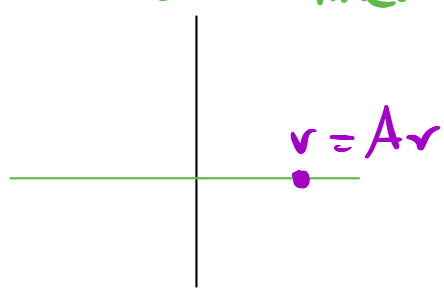
Eg: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$: a shear

Where are the eigenvectors?

- $v = \begin{pmatrix} x \\ 0 \end{pmatrix} \rightsquigarrow Av = \begin{pmatrix} x \\ 0 \end{pmatrix} = v$

  The (nonzero) vectors on the x-axis are eigenvectors with eigenvalue 1.

Av & v are on the same line.

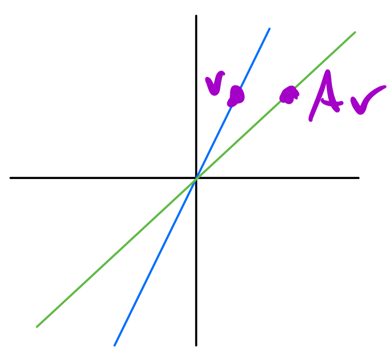


- $v = \begin{pmatrix} x \\ y \end{pmatrix}$ with $x, y \neq 0$:

  $Av = \begin{pmatrix} x+y \\ y \end{pmatrix}$

  This is not a multiple of v because $1 = \frac{y}{y} \neq \frac{x+y}{y}$.

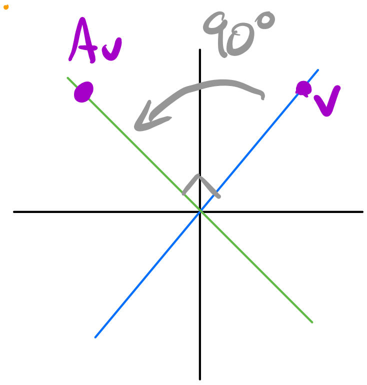
Av & v are on different lines.



[demo]

Eg: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$: CCW rotation by $90°$

There are no (real) eigenvectors: v & Av are never on the same line (unless $v=0$).



[demo]

Eigenspaces

Given an eigenvalue λ, how do you compute the λ-eigenvectors?

$$\begin{aligned} Av = \lambda v &\iff Av - \lambda v = 0 \\ &\iff Av - \lambda I_n v = 0 \\ &\iff (A - \lambda I_n)v = 0 \\ &\iff v \in \mathrm{Nul}(A - \lambda I_n) \end{aligned}$$

Def: Let λ be an eigenvalue of an $n \times n$ matrix A. The λ-eigenspace of A is

$$\begin{aligned} \mathrm{Nul}(A - \lambda I_n) &= \{v \in \mathbb{R}^n : Av = \lambda v\} \\ &= \{\text{all } \lambda\text{-eigenvectors and } 0\} \end{aligned}$$

Eg: $A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$ $\lambda = 2$

$$A - 2I_3 = \begin{pmatrix} -2 & 13 & 12 \\ 1/4 & -2 & 0 \\ 0 & 1/2 & -2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -32 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{PVF}} \mathrm{Nul}(A - 2I_3) = \mathrm{Span}\left\{ \begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix} \right\}$$

This line is the 2-eigenspace: all 2-eigenvectors are multiples of $\begin{pmatrix} 32 \\ 4 \\ 1 \end{pmatrix}$

[demo]

Eg: $A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$ $\lambda = -1$

$$A - (-1)I_3 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 2 \\ -1 & 1 & 2 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{PVF}} \mathrm{Nul}(A - (-1)I_3) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

This plane is the (-1)-eigenspace. All (-1)-eigenvectors are linear combinations of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

[demo]

NB: If λ is an eigenvalue then there are infinitely many λ-eigenvectors: the λ-eigenspace is a nonzero subspace. (This means $A - \lambda I_n$ has a free variable.)

Eg: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ $\lambda = 0 \rightsquigarrow A - \lambda I_3 = A$!

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \overset{\text{rref}}{\rightsquigarrow} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{\text{PVF}}{\rightsquigarrow} \text{Span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right\}$$

This line is the 0-eigenspace

NB: 0 is a legal eigenvalue (not an eigenvector) and

$$(0\text{-eigenspace}) = \text{Nul}(A - 0 I_n) = \text{Nul}(A)$$
$$= \{x \in \mathbb{R}^n : Ax = 0x\}$$

$$\boxed{\text{The 0-eigenspace is the null space}}$$

So if 0 is an eigenvalue of A then $\text{Nul}(A) \neq \{0\}$, so A is not invertible (not FCR).

$$\boxed{A \text{ is invertible} \iff 0 \text{ is not an eigenvalue}}$$

Eg: Let V be a subspace of $\mathbb{R}^n$, P_V the projection matrix. What are the eigenvectors & eigenvalues?

- $P_V b = 1b \iff b = b_V \iff b \in V$

  V is the 1-eigenspace

- $P_V b = 0 = 0b \iff b \in V^\perp$

  $V^\perp$ is the 0-eigenspace

[demo]

The Characteristic Polynomial

Given an eigenvalue λ of A, we know how to find all λ-eigenvectors: $\mathrm{Nul}(A-\lambda I_n)$.

How do we find the eigenvalues of A?

Eg: $A = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$ $\lambda = 1$

$$A - 1I_3 = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix} \xrightarrow{\text{ref}} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

This has full column rank: $\mathrm{Nul}(A - 1I_3) = \{0\}$.

This means 1 is not an eigenvalue of A.

Indeed, λ is an eigenvalue of A

$\iff Av = \lambda v$ has a nonzero solution v

$\iff (A - \lambda I_n)v = 0$ has a nonzero solution

$\iff \mathrm{Nul}(A - \lambda I_n) \neq \{0\}$

$\iff A - \lambda I_n$ is not invertible

$\iff \det(A - \lambda I_n) = 0$

This is an equation in λ whose solutions are the eigenvalues!

Eg: Find all eigenvalues of $A = \begin{pmatrix} 0 & 13 & 12 \\ 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}$

$$\det(A - \lambda I_3) = \det \begin{pmatrix} -\lambda & 13 & 12 \\ 1/4 & -\lambda & 0 \\ 0 & 1/2 & -\lambda \end{pmatrix}$$

$$\xrightarrow[\text{cofactors}]{\text{expand}} -\lambda \det \begin{pmatrix} -\lambda & 0 \\ 1/2 & -\lambda \end{pmatrix} - \frac{1}{4} \det \begin{pmatrix} 13 & 12 \\ 1/2 & -\lambda \end{pmatrix} + 0 \cdots$$

$$= -\lambda^3 - \frac{1}{4}(-13\lambda - 6) = -\lambda^3 + \frac{13}{4}\lambda + \frac{3}{2}$$

We need to find the zeros of a cubic polynomial:

$$p(\lambda) = -\lambda^3 + \frac{13}{4}\lambda + \frac{3}{2} = 0$$