

Complex Numbers: Crash Course

Eg: Diagonalize $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ (CCW rotation by 90°)

The characteristic polynomial is

$$p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 + 1$$

This has no real roots: $\lambda^2 + 1 = 0 \Leftrightarrow \lambda^2 = -1$.

Solution: Add a $\sqrt{-1}$ to our number system!

Def: The unit imaginary number is a number i such that $i^2 = -1$. A complex number is a number $a+bi$ for $a, b \in \mathbb{R}$.

$$\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$$

is the set of all complex numbers.

If $z = a+bi$ is a complex number, its

- real part is $\text{Re}(z) = a$, and its
- imaginary part is $\text{Im}(z) = b$.

We can add/subtract & multiply complex numbers:

$$(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$$

$$\begin{aligned} (a+bi)(c+di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Division: see p.4

Question: Wait! Why can I just declare that -1 has a square root?

Answer 1: Why can you declare that $\sqrt{2}$ has a square root? You can't write it down—it's an infinite non-repeating decimal...

Answer 2: Take Math 401/501 for a systematic treatment.

Complex numbers have an additional algebraic operation.

Def: The **complex conjugate** of $z = a+bi$ is

$$\bar{z} = a - bi$$

(replace i by $-i$ = the other $\sqrt{-1}$)

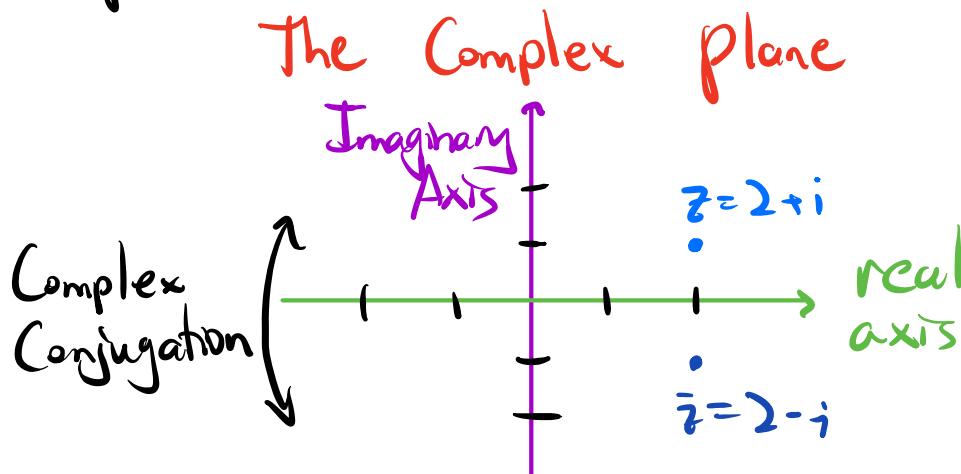
Check: $\overline{(z+w)} = \bar{z} + \bar{w}$ $\overline{zw} = \bar{z} \cdot \bar{w}$ $\overline{\bar{z}} = z$

NB: if $z = a+bi$ then

- $z + \bar{z} = (a+bi) + (a-bi) = 2a = 2\operatorname{Re}(z)$
- $z - \bar{z} = (a+bi) - (a-bi) = 2bi = 2i\operatorname{Im}(z)$

$$z + \bar{z} = 2\operatorname{Re}(z) \quad z - \bar{z} = 2i\operatorname{Im}(z)$$

Since a complex number $z = a+bi$ is determined by two real numbers a and b , we can draw \mathbb{C} as a plane:



Complex conjugation negates the imaginary coordinate:
it **flips** over the real axis.

NB: A real number is also a complex number:
 $a \in \mathbb{R} \rightsquigarrow a+0i \in \mathbb{C}$

So \mathbb{C} contains \mathbb{R} . **NB:** $z \in \mathbb{R} \iff z = \bar{z}$

NB: If $z = a+bi$ then

$$z\bar{z} = (a+bi)(a-bi) = a^2 - b^2i^2 = a^2 + b^2$$

nonnegative
real
number

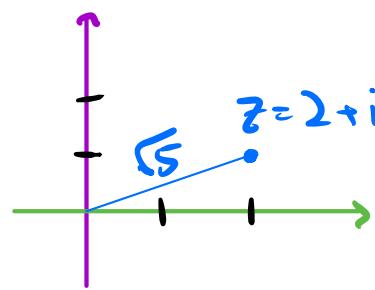
$$z = a+bi \rightsquigarrow z\bar{z} = a^2 + b^2 \geq 0$$

Def: The **modulus** of z is $|z| = \sqrt{z\bar{z}}$

This is its **length** as a vector in the complex plane.

Eg: $z = 2+i$

$$\rightarrow |z| = \sqrt{4+1} = \sqrt{5}$$



Eg: If $a \in \mathbb{R}$ then $a = a + 0i$ and

$$|a| = \sqrt{a^2 + 0^2} = \sqrt{a^2} = |a|$$

(the modulus is the usual absolute value)

Check: $|zw| = |z| \cdot |w|$ $|\bar{z}| = |z|$

Here's how to take a reciprocal of $z = a+bi \neq 0$:

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2} \leftarrow \text{positive real number}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \text{or} \quad \frac{1}{a+bi} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Check: $\left| \frac{1}{z} \right| = \frac{1}{|z|}$

Eg: $\frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5}i$

Check: $(2+i)\left(\frac{2}{5} - \frac{1}{5}i\right) = \frac{4}{5} + \frac{1}{5} + \left(\frac{2}{5} - \frac{2}{5}\right)i = 1 \checkmark$

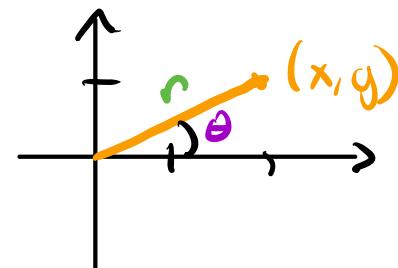
Now we can divide too: $\frac{w}{z} = w \cdot \frac{1}{z} = \frac{w\bar{z}}{|z|^2}$.

Polar Coordinates

Recall that a point in the (x,y) -plane can be specified in **polar coordinates** (r, θ) :

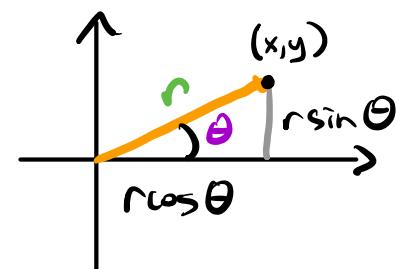
$$r = \text{length of } (\vec{x}) = \sqrt{x^2 + y^2}$$

$$\theta = \text{angle } (\vec{x}) \text{ makes with the positive } x\text{-axis} = \pm \arctan(\frac{y}{x})$$



To go from polar coordinates back to Cartesian (x,y) -coordinates:

$$(r, \theta) \rightarrow x = r \cos \theta \\ y = r \sin \theta$$



If we apply this to a complex number $z = a+bi$:

$$r = \sqrt{a^2 + b^2} = |z| \rightarrow a = |z| \cos \theta \quad b = |z| \sin \theta$$

$$\Rightarrow z = |z| (\cos \theta + i \sin \theta)$$

Def: The **argument** of $z = a+bi$ is

$\arg(z) = \theta =$ the angle z makes with the positive real axis.

So you can specify a complex number in 2 ways:
 (Cartesian coords / real & imaginary parts)
 $z = a + bi$

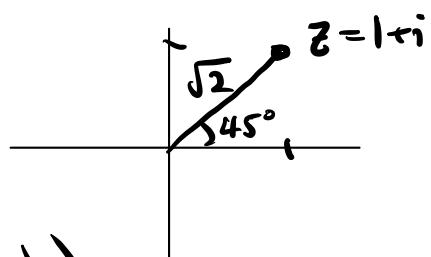
(Polar coords / modulus & argument)

$$z = |z|(\cos \theta + i \sin(\theta)) \quad \theta = \arg(z)$$

Eg: $z = 1+i \rightarrow |z| = \sqrt{1+1} = \sqrt{2}$

$$\arg(z) = 45^\circ$$

$$\rightarrow z = \sqrt{2} (\cos(45^\circ) + i \sin(45^\circ))$$

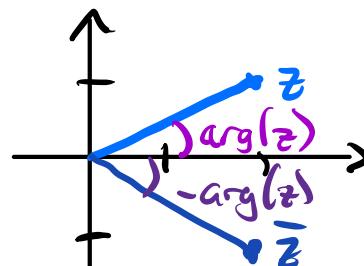


Check: $\cos(45^\circ) = \sin(45^\circ) = 1/\sqrt{2}$

$$z = 1+i = \sqrt{2} (1/\sqrt{2} + i/\sqrt{2}) \quad \checkmark$$

Facts:

- $\arg(\bar{z}) = -\arg(z)$
 (flip over real axis)



- $\arg(1/z) = -\arg(z)$
 ($1/z = \bar{z}/|z|^2 = (\text{positive number}) \cdot \bar{z}$)

Fact: $\arg(zw) = \arg(z) + \arg(w)$

Proof: This is a trig identity!

$$z = |z|(\cos \theta + i \sin \theta) \quad \theta = \arg(z)$$

$$w = |w|(\cos \varphi + i \sin \varphi) \quad \varphi = \arg(w)$$

$$zw = |z||w|(\cos \theta + i \sin \theta)(\cos \varphi + i \sin \varphi)$$

$$= |zw|(\cos \theta \cos \varphi - \sin \theta \sin \varphi + (\cos \theta \sin \varphi + \sin \theta \cos \varphi)i)$$

$$= |zw|(\cos(\theta + \varphi) + i \sin(\theta + \varphi))$$

$$\Rightarrow \theta + \varphi = \arg(zw)$$

sum angle
formulas



We like polar coordinates because multiplication is easier!

$$z = |z|(\cos \theta + i \sin \theta) \quad w = |w|(\cos \varphi + i \sin \varphi)$$

$$\Rightarrow \begin{cases} zw = |z||w|(\cos(\theta + \varphi) + i \sin(\theta + \varphi)) \\ \bar{z} = |z|(\cos \theta - i \sin \theta) \\ \frac{1}{z} = \frac{1}{|z|}(\cos \theta - i \sin \theta) \end{cases} \quad \begin{cases} \cos(-\theta) = \cos \theta \\ \sin(-\theta) = -\sin \theta \end{cases}$$

Exercise: expand the product

$$(\cos \theta + i \sin \theta)^3 = \cos(3\theta) + i \sin(3\theta)$$

to derive the triple-angle formulas.

Euler's Formula: For any real number θ ,

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Proof: Take Taylor expansions of both sides...

Polar Form of Complex Numbers, Alternative:

$$z = |z| e^{i\theta} \quad \theta = \arg(z)$$

This makes the formulas on p. 7 easier to remember:

$$zw = (|z| e^{i\theta})(|w| e^{i\phi}) = |z||w| e^{i(\theta+\phi)}$$

$$\bar{z} = |z| e^{-i\theta}$$

$$\frac{1}{z} = (|z| e^{i\theta})^{-1} = \frac{1}{|z|} e^{-i\theta}$$

Eg: $-1 = e^{i\pi}$ $\frac{1}{\sqrt{2}}(1+i) = e^{i\pi/4}$

It turns out that once $x^2+1=0$ has a solution, then any polynomial has a root!

Fundamental Theorem of Algebra:

Every polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$a_0, a_1, \dots, a_n \in \mathbb{C} \quad a_n \neq 0$$

can be factored into linear terms:

$$p(x) = a_n (x - \lambda_1) \cdots (x - \lambda_n) \quad \lambda_1, \dots, \lambda_n \in \mathbb{C}$$

Eg: $p(x) = x^2 + x + 1$

Use the quadratic formula:

$$x = \frac{1}{2} \left(-1 \pm \sqrt{1-4} \right) = \frac{1}{2} \left(-1 \pm i\sqrt{3} \right)$$

$\sqrt{-3} = \sqrt{-1 \cdot 3} = \sqrt{-1} \cdot \sqrt{3} = i\sqrt{3}$

$$\text{So } p(x) = (x - \frac{1}{2}(-1 + i\sqrt{3}))(x - \frac{1}{2}(-1 - i\sqrt{3}))$$

Eg: $p(x) = x^2 + 1 = (x+i)(x-i)$

So now $\begin{bmatrix} 0 & -1 \\ i & 0 \end{bmatrix}$ has two eigenvalues $\pm i$.

Real Polynomials:

If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has real coefficients $a_0, a_1, \dots, a_n \in \mathbb{R}$

then its complex roots come in conjugate pairs:

$$p(\lambda) = 0 \iff p(\bar{\lambda}) = 0$$

Check:

$$\begin{aligned} p(\bar{\lambda}) &= a_n \bar{\lambda}^n + a_{n-1} \bar{\lambda}^{n-1} + \dots + a_1 \bar{\lambda} + a_0 \\ (\bar{a}_i = a_i) &= \overline{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0} \\ &= \overline{p(\lambda)} \end{aligned}$$

$$\text{So } p(\lambda) = 0 \iff p(\bar{\lambda}) = 0.$$