Geometry of Diagonalizable Matrices

Last time: an $n\times n$ matrix A is **diagonalizable** if it has an **eigenbasis** $\{w_1,\ldots,w_n\}$.

Matrix form: $A = CDC^{-1}$ where

$$C = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

In this case every vector can be written as a linear combination of eigenvectors, so the action of A on $\mathbb{R}^n$ is reduced to **scalar multiplication**:

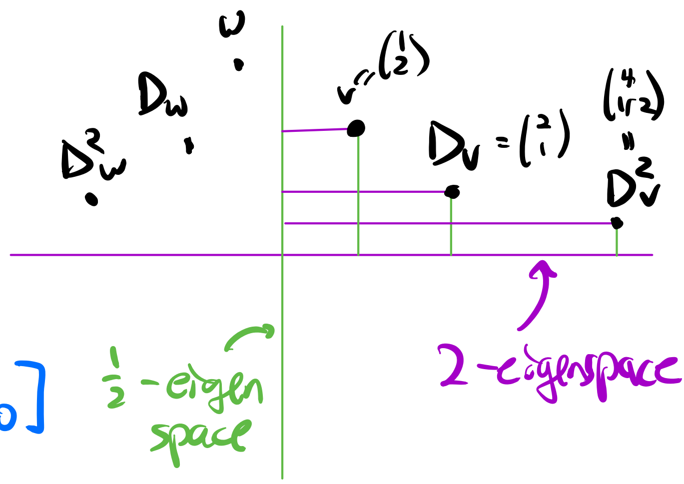
$$A(x_1w_1 + \cdots + x_nw_n) = x_1\lambda_1w_1 + \cdots + x_n\lambda_nw_n.$$

What does this mean **geometrically?**

→ If you work in an eigenbasis, A acts like a **diagonal matrix**.

Eg: $D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ so $D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix}$.

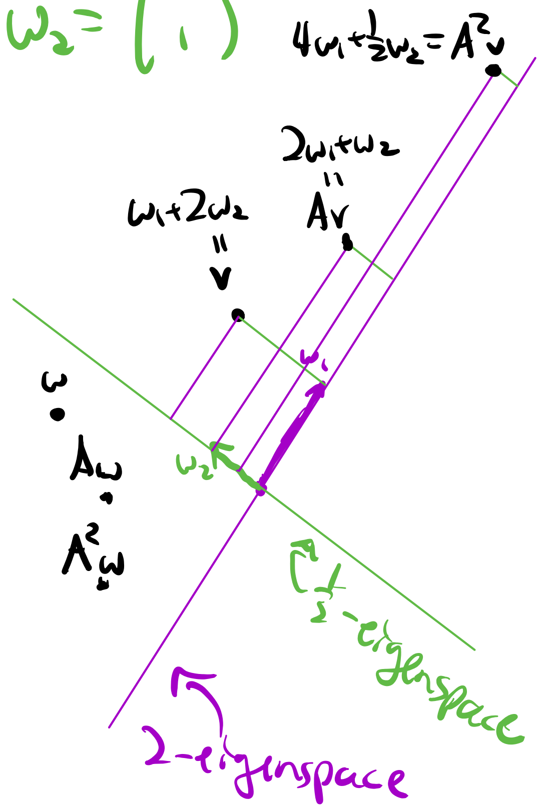
- scales the x-direction by 2
- scales the y-direction by $\frac{1}{2}$



[demo]

Eg: $A = \frac{1}{10}\begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - \frac{5}{2}\lambda + 1 = (\lambda - 2)(\lambda - \frac{1}{2})$

$\lambda_1 = 2 \quad w_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



Work in the eigenbasis!
(think in terms of LCs of w_1, w_2)

$A(x_1 w_1 + x_2 w_2) = 2x_1 w_1 + \frac{1}{2} x_2 w_2$

- scales the w_1-direction by 2
- scales the w_2-direction by $1/2$

[demo]

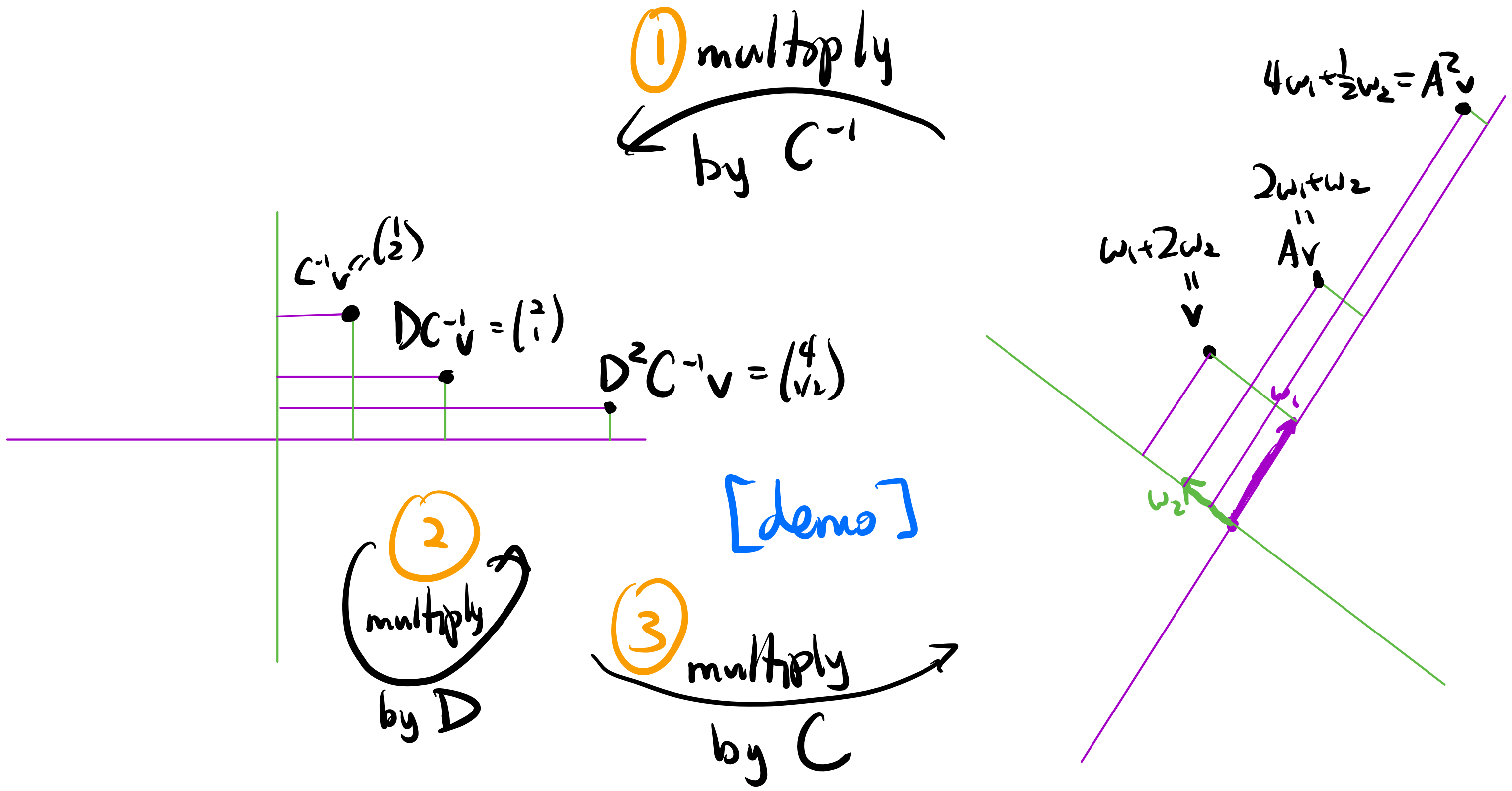
This is the vector form. In matrix form

$A = CDC^{-1} \qquad C = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$

Then $Av = CDC^{-1}v$

= " • first multiply v by C^{-1}
• then multiply by the diagonal matrix D
• then multiply by C again"

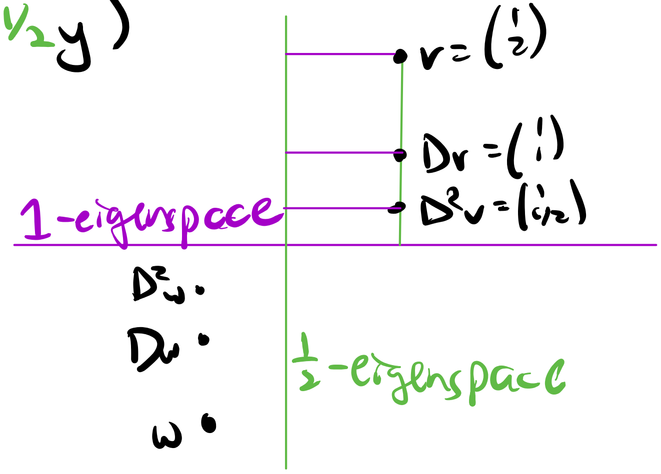
Note $C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 w_1 + x_2 w_2 \iff C^{-1}(x_1 w_1 + x_2 w_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$



Eg: $D = \begin{pmatrix}1 & 0\\0 & 1/2\end{pmatrix}$ $\quad D\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\ \frac{1}{2}y\end{pmatrix}$

- scales the x-direction by 1
- scales the y-direction by $1/2$

[demo]



Eg: $A = \frac{1}{6}\begin{pmatrix}5 & 1\\2 & 4\end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)(\lambda-\frac{1}{2})$

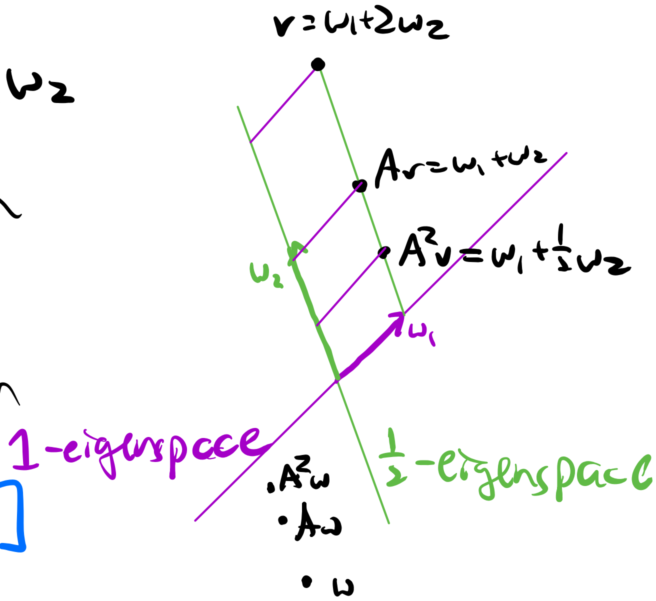
$\lambda_1 = 1 \quad w_1 = \begin{pmatrix}1\\1\end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad w_2 = \begin{pmatrix}-1\\2\end{pmatrix}$

Work in the eigenbasis!

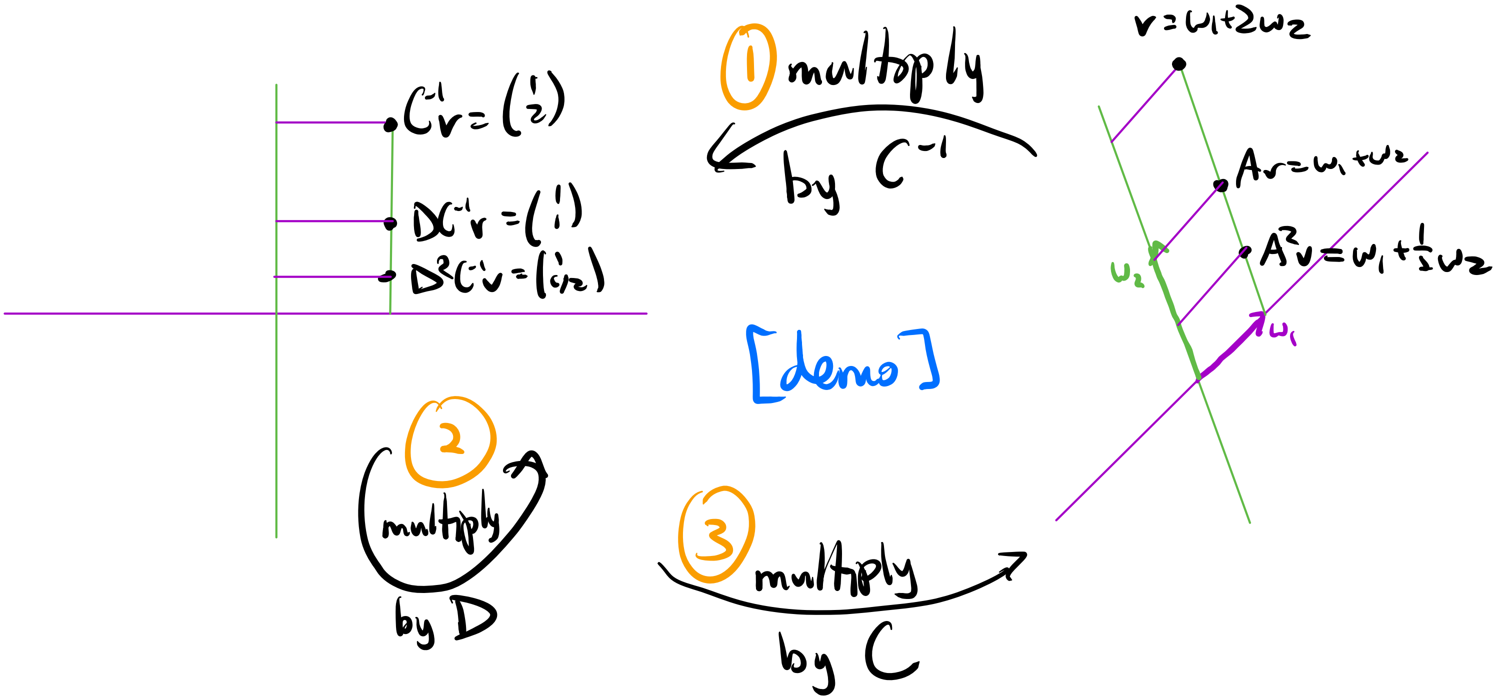
$$A(x_1w_1 + x_2w_2) = 1\,x_1w_1 + \tfrac{1}{2}\,x_2w_2$$

- scales the w_1-direction by 1
- scales the w_2-direction by $1/2$

[demo]



Matrix Form: $A = CDC^{-1}$ $\quad C = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$



Eg: $A = \dfrac{1}{580}\begin{pmatrix} 503 & 73 & 269 \\ 207 & 1137 & -49 \\ 270 & -30 & 680 \end{pmatrix}$ has eigenbasis

$$w_1 = \begin{pmatrix} -7 \\ 2 \\ 5 \end{pmatrix} \quad w_2 = \begin{pmatrix} -1 \\ -9 \\ 0 \end{pmatrix} \quad w_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

and eigenvalues

$$\lambda_1 = 1/2 \quad \lambda_2 = 2 \quad \lambda_3 = 3/2$$

Work in the eigenbasis!

$$A(x_1w_1 + x_2w_2 + x_3w_3) = \frac{1}{2}x_1w_1 + 2x_2w_2 + \frac{3}{2}x_3w_3$$

- scales the w_1-direction by $\frac{1}{2}$
- scales the w_2-direction by 2
- scales the w_3-direction by $\frac{3}{2}$

[demo]

Diagonalization with Complex Eigenvalues

Diagonalization still works great even if the eigenvalues are not real.

→ Still can solve difference equations & ODEs

→ Still get real-number answers

Fact: The complex eigenvalues & eigenvectors of a real matrix come in complex conjugate pairs:

$$Av = \lambda v \iff A\bar{v} = \bar{\lambda}\bar{v}$$

here $v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \longrightarrow \bar{v} = \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}$

Eg: Solve the difference equation

$$v_{n+1} = A v_n \qquad A = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \qquad v_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

No complex #s in the statement!

(ie find $A^k v_0$ for all k).

We try diagonalization:

$$p(\lambda) = \lambda^2 + 3\lambda + 3 \qquad \lambda = \frac{1}{2}\left(-3 \pm \sqrt{9-12}\right)$$

$$\to \lambda = \frac{1}{2}\left(-3 + i\sqrt{3}\right), \quad \bar{\lambda} = \frac{1}{2}\left(-3 - i\sqrt{3}\right)$$

Find eigenvectors using the 2×2 trick (HW10 #3)

$$w = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2}(3 - i\sqrt{3}) \end{pmatrix} \qquad \bar{w} = \begin{pmatrix} 1 \\ \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix}$$

eigenvector for λ — eigenvector for $\bar{\lambda}$

Check: $$Aw = \begin{pmatrix} -\frac{1}{2}(3 - i\sqrt{3}) \\ 3 - \frac{3}{2}(3 - i\sqrt{3}) \end{pmatrix} = \begin{pmatrix} -3/2 + \frac{1}{2} i\sqrt{3} \\ -3/2 + \frac{3}{2} i\sqrt{3} \end{pmatrix}$$

Wait, is this a multiple of v??

$$\frac{1}{2}\left(-3 + i\sqrt{3}\right) w = \frac{1}{2}\left(-3 + i\sqrt{3}\right) \begin{pmatrix} 1 \\ \frac{1}{2}(3 - i\sqrt{3}) \end{pmatrix}$$

$$= \begin{pmatrix} -3/2 + \frac{1}{2} i\sqrt{3} \\ \frac{1}{4}\left(-9 + 3 + i(3\sqrt{3} + 3\sqrt{3})\right) \end{pmatrix} = \begin{pmatrix} -3/2 + 1/2\, i\sqrt{3} \\ -3/2 + 3/2\, i\sqrt{3} \end{pmatrix} \checkmark$$

equal

So $A = CDC^{-1}$ for

$$C = \begin{pmatrix} 1 & 1 \\ \frac{1}{2}(3 - i\sqrt{3}) & \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix} \quad D = \begin{pmatrix} \frac{1}{2}(-3 + i\sqrt{3}) & 0 \\ 0 & \frac{1}{2}(-3 - i\sqrt{3}) \end{pmatrix}$$

To compute $v_k = A^k v_0$, we need to solve

$$v_0 = x_1 w + x_2 \bar{w}$$

Eyeball it: $w + \bar{w} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = v_0$ ✓

$$\Rightarrow A^k v_0 = \lambda^k w + \bar{\lambda}^k \bar{w}$$

$$= \left(\frac{1}{2}(-3 + i\sqrt{3})\right)^k \begin{pmatrix} 1 \\ \frac{1}{2}(3 - i\sqrt{3}) \end{pmatrix} + \left(\frac{1}{2}(-3 - i\sqrt{3})\right)^k \begin{pmatrix} 1 \\ \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix}$$

So far it's exactly the same as for real eigenvalues!

... but we wanted a solution involving only real #s.

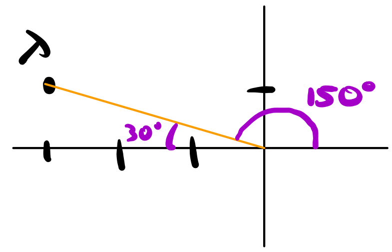
Thankfully, $\lambda^k w$ and $\bar{\lambda}^k \bar{w}$ are complex conjugates, so

$$A^k v_0 = \lambda^k w + \bar{\lambda}^k \bar{w} = 2\mathrm{Re}[\lambda^k w]$$

$$= 2\mathrm{Re} \begin{pmatrix} [\frac{1}{2}(-3 + i\sqrt{3})]^k \\ -[\frac{1}{2}(-3 + i\sqrt{3})]^{k+1} \end{pmatrix} = 2\mathrm{Re} \begin{pmatrix} \lambda^k \\ -\lambda^{k+1} \end{pmatrix}$$

Recall: Multiplication of complex numbers is much easier in polar form.

$$\lambda = \tfrac{1}{2}\left(-3 + i\sqrt{3}\right) = r e^{i\theta}$$



$$r = \tfrac{1}{2}\sqrt{9+3} = \tfrac{1}{2}\sqrt{4\cdot 3} = \sqrt{3}$$

$$\theta = 150^\circ = 5\pi/6$$

So $\lambda^k = r^k e^{ik\cdot\frac{5\pi}{6}} = (\sqrt{3})^k\left(\cos\frac{5k\pi}{6} + i\sin\frac{5k\pi}{6}\right)$

$$\Rightarrow \operatorname{Re}(\lambda^k) = (\sqrt{3})^k \cos\frac{5k\pi}{6}$$

$$\Rightarrow \boxed{v_k = 2(\sqrt{3})^k \begin{pmatrix} \cos(5k\pi/6) \\ \cos(5(k+1)\pi/6) \end{pmatrix}}$$

[demo]

The answer involves only real numbers (and cosines-weird!) but we needed complex numbers to get it!

Difference Equations with Complex Eigenvalues

To compute $A^k v_0$ for any k (A a real matrix):

(1) Find all eigenvalues & eigenvectors of A, including complex ones.
→ Remember $Av = \lambda v \iff A\bar{v} = \bar{\lambda}\bar{v}$

(2) If you end up with $< n$ eigenvectors, the matrix is not diagonalizable. Otherwise, let $w_1, \dots, w_n$ be a (complex) eigenbasis.

(3) Solve $v_0 = x_1 w_1 + \cdots + x_n w_n$. Then

$$A^k v_0 = \lambda_1^k x_1 w_1 + \cdots + \lambda_n^k x_n w_n$$

(4) To get an answer with only real numbers, group complex conjugate pairs:

$$\lambda^k x w + \bar{\lambda}^k \bar{x} \bar{w} = 2\,\mathrm{Re}(\lambda^k x w)$$

then use polar form for λ:

$$\lambda = re^{i\theta} \rightsquigarrow \lambda^k = r^k e^{ik\theta}$$

You get an answer involving cosines & sines.