Algebraic & Geometric Multiplicity

Recall: We like diagonalizable matrices because then we can solve difference equations:

- if $\{w_1, \dots, w_n\}$ is a basis of $\mathbb{R}^n$ of eigenvectors then any $v_0 \in \mathbb{R}^n$ is a linear combination of w_i's

$$v_0 = x_1 w_1 + \cdots + x_n w_n$$

$$\Rightarrow A^k v_0 = \lambda^k x_1 w_1 + \cdots + \lambda^k x_n w_n$$

Today we will discuss a criterion for diagonalizability.

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

So the eigenvalues are 1 an 2.

- $\lambda = 1$: $\mathrm{Nul}(A - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

  this is a line: dimension 1

- $\lambda = 2$: $\mathrm{Nul}(A - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}\right\}$

  this is a line: dimension 1

[demo]

This matrix is not diagonalizable:

only two linearly independent eigenvectors.

Eg: $B = \begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

So the eigenvalues are 1 an 2.

- $\lambda = 1$: $\mathrm{Nul}(B - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

  this is a line: dimension 1

- $\lambda = 2$: $\mathrm{Nul}(B - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$

  this is a plane: dimension 2

This matrix is diagonalizable: $B = CDC^{-1}$ for

$$C = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 4 & 2 \\ 1 & 3 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$ [demo]

Both matrices have only 2 eigenvalues.
The difference is that B had two LI 2-eigenvectors and A had one.

Recall: If λ is a root of a polynomial $p(x)$, its multiplicity is the largest power of $(x-\lambda)$ dividing p.

Def: Let A be an $n\times n$ matrix with eigenvalue λ.

(1) The algebraic multiplicity (AM) of λ is its multiplicity as a root of the characteristic polynomial $p(\lambda)$.

(2) The geometric multiplicity (GM) of λ is the dimension of the λ-eigenspace:

$$\begin{aligned} GM(\lambda) &= \dim \operatorname{Nul}(A-\lambda I_n) \\ &= \#\text{free variables in } A-\lambda I_n. \\ &= \#\text{linearly independent } \lambda\text{-eigenvectors} \end{aligned}$$

Thm (AM $\geq$ GM): For any eigenvalue λ of A,

(algebraic multiplicity of λ)

$\geq$ (geometric multiplicity of λ) ≥ 1

NB: $GM \geq 1$ just says every eigenvalue has an eigenvector — the eigenspace can't be $\{0\}$, so its dimension is ≥ 1.

NB: This is one of the few facts I can't prove for you in the notes.

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

- $\lambda = 1$: $\mathrm{Nul}(A - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is a line.

  $AM = 1 \geq GM = 1 \geq 1$

- $\lambda = 2$: $\mathrm{Nul}(A - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}\right\}$ is a line

  $AM = 2 \geq GM = 1 \geq 1$

Eg: $B = \begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

- $\lambda = 1$: $\mathrm{Nul}(A - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ is a line.

  $AM = 1 \geq GM = 1 \geq 1$

- $\lambda = 2$: $\mathrm{Nul}(A - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}\right\}$ is a line

  $AM = 2 \geq GM = 2 \geq 1$

Upshot: if $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$ then

- the 1-eigenspace is necessarily a line:

  $AM = 1 \geq GM \geq 1$

- the 2-eigenspace is a line or a plane:

  $AM = 2 \geq GM \geq 1$

If this matrix is going to be diagonalizable, you need 3 LI eigenvectors. This means

$$GM(1) + GM(2) = 3.$$

Since $p(\lambda) = -(\lambda-2)^{2}(\lambda-1)^{1}$ has degree 3, we have

$$AM(1) + AM(2) = 2 + 1 = 3$$

Hence the matrix is only diagonalizable if

$$AM(1) = GM(1) \ \& \ AM(2) = GM(2).$$

NB: Any $p(\lambda) = (-1)^n(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_r)^{m_r}$ factors into linear factors, where $m_i = AM(\lambda_i)$. Hence

$$\boxed{AM(\lambda_1) + \cdots + AM(\lambda_r) = n \qquad \text{(sum of the AM's is } n\text{)}}$$

NB: This also holds for complex eigenvalues.

If A is diagonalizable then it has n LI eigenvectors, so

$$\begin{array}{ccccc} n = & GM(\lambda_1) & + \cdots + & GM(\lambda_n) & \\ & \wedge\!| & & \wedge\!| & \\ & AM(\lambda_1) & + \cdots + & AM(\lambda_n) & = n \end{array}$$

This forces $AM(\lambda_i) = GM(\lambda_i)$, so we've shown:

Thm (AM/GM Criterion for Diagonalizability):
Let A be an $n\times n$ matrix.

- A is diagonalizable over the complex numbers $\iff AM(\lambda)=GM(\lambda)$ for every eigenvalue λ
- A is diagonalizable over the real numbers $\iff AM(\lambda)=GM(\lambda)$ for every eigenvalue λ and A has no complex eigenvalues.

Eg: $B=\begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ is not diagonalizable because $AM(2)=2\neq 1=GM(2)$

Corollary: If A has n different eigenvalues then A is diagonalizable.

Proof: If A has n different eigenvalues then

$$n=AM(\lambda_1)+\cdots+AM(\lambda_n) \Rightarrow AM(\lambda_i)=1$$
$$1=AM(\lambda_i)\geq GM(\lambda_i)\geq 1 \Rightarrow AM(\lambda_i)=GM(\lambda_i)=1 \checkmark$$

Eg: A 2×2 real matrix with a complex eigenvalue λ is diagonalizable (over $\mathbb{C}$): it has 2 eigenvalues λ and $\bar{\lambda}$.

Differential Equations & Difference Equations

So far, our major application of eigenvalues & diagonalization has been to solving difference equations:

compute $v_k = A^k v_0$ for any k.

There is a completely analogous application to systems of ordinary differential equations:

find $u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$ such that $\frac{du}{dt} = Au$ & $u(0) = u_0$.

We'll spend the rest of this lecture discussing how to get ODE's from difference equations.

One-Dimensional Example

I have a savings account that earns 2% interest per year (I wish!) and has \$10,000 in it this year. How much will I have after k years? Let v_n = amount in year n (a scalar).

$$V_{k+1} = 1.02 \cdot V_k \qquad (= v_k + 2\% \text{ of } v_k)$$

So $v_1 = 1.02\, v_0$, $V_2 = 1.02\, v_1 = (1.02)^2 v_0$, ...,

$$V_k = (1.02)^k V_0$$

1x1 state change matrix

(think: (1) is an "eigenvector" of (1.02) ...)

Actually my interest is compounded monthly.

This means every month I earn $\frac{2\%}{12}$ interest.

$$V_{k+\frac{1}{12}} = \left(1+\frac{0.02}{12}\right)V_k$$

$$V_{1/12} = \left(1+\frac{0.02}{12}\right)V_0 \qquad V_{2/12} = \left(1+\frac{0.02}{12}\right)^2 V_0, \ \ldots$$

$$\rightsquigarrow V_k = V_{\frac{12k}{12}} = \left[\left(1+\frac{.02}{12}\right)^{12}\right]^k V_0$$

$$\approx (1.02018)^k V_0 \quad \rightsquigarrow 2.018\% \text{ APR}$$

What if it's compounded daily?

$$V_k = \left[\left(1+\frac{.02}{365}\right)^{365}\right]^k V_0$$

$$\approx (1.0202007)^k V_0 \quad \rightsquigarrow 2.02007\% \text{ APR}$$

What if it's compounded every minute?

$365 \times 24 \times 60 = 525{,}600$ minutes per year

$$V_k = \left[\left(1+\frac{.02}{525{,}600}\right)^{525{,}600}\right]^k V_0$$

$$\approx (1.0202013)^k V_0 \quad \rightsquigarrow 2.02013\% \text{ APR}$$

What if it is continuously compounded?

$$\lim_{p\to\infty}\left(1+\frac{.02}{p}\right)^p = ?$$

Theorem: $\lim_{p\to\infty}\left(1+\frac{\lambda}{p}\right)^p = e^\lambda$

So for continuously compounded interest,

$$v_k = \left(e^{0.02}\right)^k v_0 \qquad e^{0.02} \simeq 1.0202013400$$

More generally, if t is any real number (like 1.35 years),

$$v_t = e^{0.02t} v_0$$

This solves the ordinary differential equation

$$u'(t) = 0.02u(t) \qquad u(0) = u_0 = 10{,}000$$

(Makes sense: it says the \$ in my savings account is increasing at a rate of 2% /year times the amount in the account at the moment.)

This is the 1-D situation. Increase the dimension:

- $v_{k+1} = (1+\lambda)v_k \rightsquigarrow$ difference equation $v_{k+1} = Av_k$
- $u' = \lambda u \rightsquigarrow$ System of ODEs $u' = Au$

There is a similar relationship between difference equations & systems of ODEs:

$$\boxed{\begin{array}{c}\text{Difference equations} \longrightarrow \text{systems of ODEs}\\ \text{as the sampling time} \longrightarrow 0\end{array}}$$