# Recorded Lecture: Basic Definitions

This is recorded because the material is straightforward

$\rightsquigarrow$ more fun in $1^{st}$ lecture

**Scalars:** are (real) numbers (complex in week 11)

Notation: $c \in \mathbb{R}$ "$\in$" means "is an element of"
"$\mathbb{R}$" means "all real numbers"

"Example"

Eg: $c = 7, -\pi, 2^e, 0, 1, \ldots$

**Vectors:** are an (ordered) list of (finitely many) real numbers.

Notation: $v \in \mathbb{R}^n$ "$\mathbb{R}^n$" means "lists of $n$ numbers"
or "vectors of size $n$"

Eg: $\underbrace{v = \begin{pmatrix} 7 \\ -\pi \\ 2^e \end{pmatrix}}_{}$ size 3, $v = \begin{pmatrix} 7 \\ -\pi \\ 2^e \end{pmatrix} \in \mathbb{R}^3$ size 4, $v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4$

= coordinates/entries

Note: We usually write a vector as a column ("column vector") but that's just notation: $v = (7, -\pi, 2^e)$ means the same thing.

**Note:** I won't decorate vectors like $\vec{v}$ or $\mathbf{v}$ because it's annoying.

**Important eg:** The unit coordinate vectors are

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^n$$

$e_i =$ the vector with a $1$ in the $i^{\text{th}}$ entry and zeros elsewhere

This notation is fixed for the whole semester.

It is ambiguous: have to know $n$ from context.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$n = 3 \qquad\qquad n = 2$$

**Sub-eg:** $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$

these are the unit coordinate vectors in $\mathbb{R}^3$.

**Eg:** The zero vector is $0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$

$\rightarrow$ again have to know $n$ from context.

Two vectors are equal if they have the same size and coordinates.

**Eg:** $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

# Vector Algebra

## Scalar Multiplication:

scalar × vector ⇝ vector

$c \in \mathbb{R}$ scalar $\quad v \in \mathbb{R}^n$ vector $\rightsquigarrow$ $cv \in \mathbb{R}^n$

$$c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix}$$

Eg:
$$2\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \qquad 0\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$(-1)\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

## Vector Addition/Subtraction:

vector ± vector ⇝ vector

$v, w \in \mathbb{R}^n \rightsquigarrow v+w \in \mathbb{R}^n \quad v-w \in \mathbb{R}^n$

Compute coordinate-wise:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1-y_1 \\ \vdots \\ x_n-y_n \end{pmatrix}$$

"Note"

**NB:** Only makes sense to add/subtract vectors of the same size.

Eg:
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \times$$

# Dot Product / Inner Product:

vector × vector ⇝ scalar

multiply components & add:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

$v, w \in \mathbb{R}^n \rightsquigarrow v\cdot w \in \mathbb{R}$

Eg: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32 \in \mathbb{R}$

NB: Only makes sense to take the dot product of vectors of the same size.

## Rules for vector algebra:

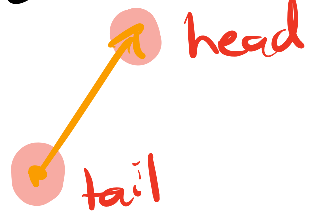
- $c(v \pm w) = cv \pm cw$ — distributivity
- $v\cdot(cw) = c(v\cdot w) = (cv)\cdot w$ — distributivity
- $w\cdot v = v\cdot w$ — commutativity
- $u\cdot(v \pm w) = u\cdot v \pm u\cdot w$ — distributivity
- $\cancel{u\cdot(v\cdot w) = (u\cdot v)\cdot w}$

  ↳ not defined: $v\cdot w$ is a scalar

# Vector Geometry

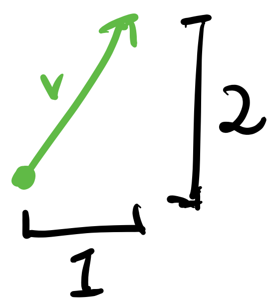
→ will be very important!!

We will usually draw a vector as an arrow

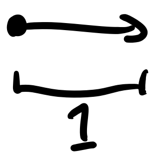



Components of a vector record displacements along each axis.

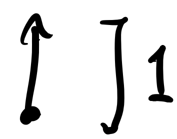
Eg: $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \mathbb{R}^2$




$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$




$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$




## Scalar Multiplication:

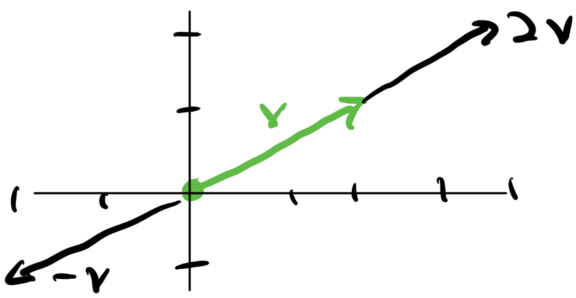
length of $c \cdot v$ = $|c| \cdot$ length of $v$

changes the length of a vector:

direction is unchanged (for scalars $> 0$)

direction is reversed (for scalars $< 0$)

Eg: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$




$2v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$-v = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

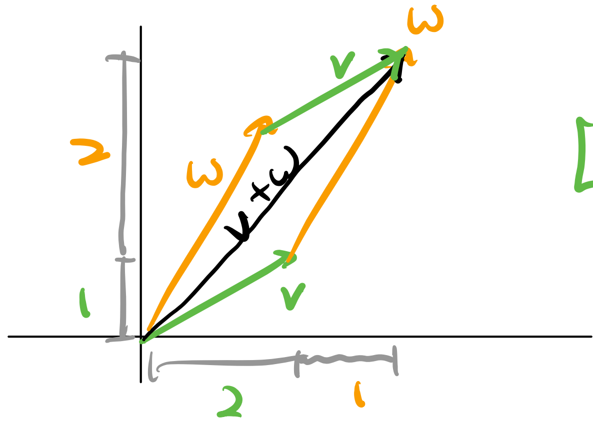
**Vector Addition:** Adds displacements

**Paralellogram Law:** to draw $v+w$, draw the tail of $v$ at the head of $w$ (or vice-versa); the head of $v$ is at $v+w$.

Eg: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$v+w = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$



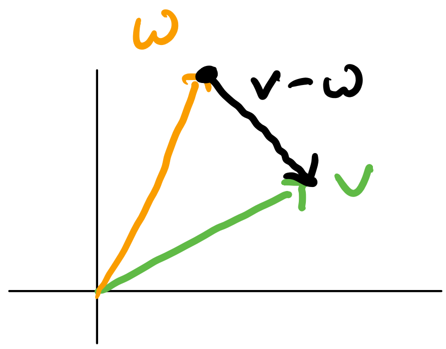

[demo]

**Vector Subtraction:** $w + (v-w) = v$

Eg: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$v-w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$




**Linear Combinations:** combine addition & scalar ×.

$v_1, \ldots, v_r \in \mathbb{R}^n \quad c_1, \ldots, c_r \in \mathbb{R} \rightsquigarrow$

$$c_1 v_1 + c_2 v_2 + \cdots + c_r v_r \in \mathbb{R}^n$$

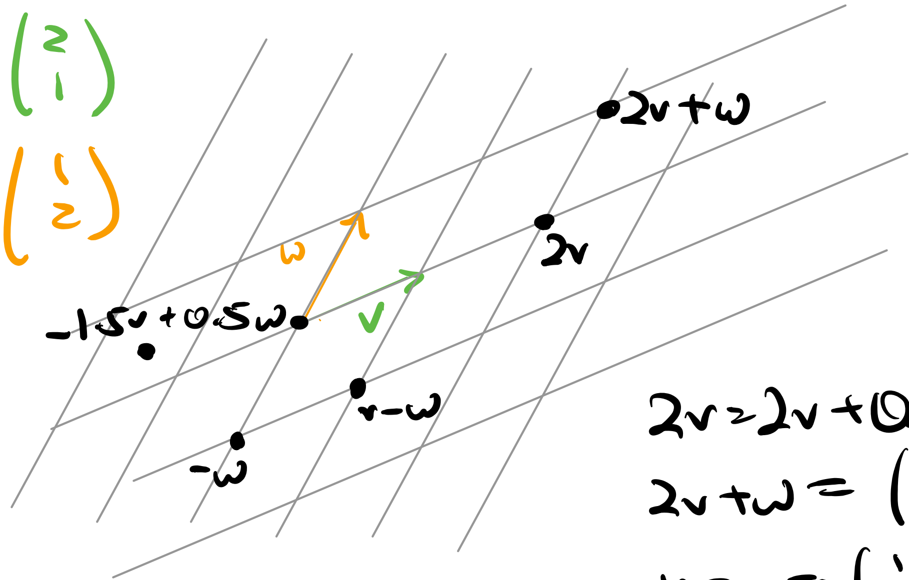
↳ coefficients/weights of the linear comb.

$$v_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$c_1 v_1 + c_2 v_2 = c_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c_2 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 x_1 + c_2 y_1 \\ c_1 x_2 + c_2 y_2 \end{pmatrix}$$

Eg: $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$




$2v = 2v + 0w = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$2v + w = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$v - w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Like giving directions:

"To get to $-1.5v + 0.5w$,
first go $1.5 \times$ length of $v$ in opposite $v$-direction, then go $0.5 \times$ length of $w$ in $w$-direction"

# Dot Product: lengths & angles

→ this is how to do problems involving lengths & angles

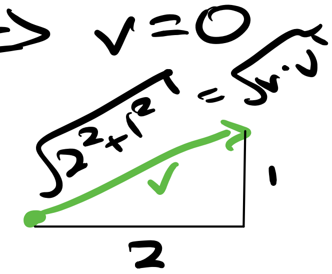
**Special case:** dot a vector with itself.

$$v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$$

$$\cancel{v^2} = v \cdot v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 x_1 + x_2 x_2 = x_1^2 + x_2^2 \geq 0$$

$$v \cdot v = 0 \iff v = 0$$

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$




**Pythagorean Theorem:** length of $v$ is $\sqrt{v \cdot v} = \sqrt{x_1^2 + x_2^2}$

**Def:** The length of a vector $v$ is

$$\|v\| := \sqrt{v \cdot v} \qquad \|v\|^2 = v \cdot v$$

"defined to be"

**Eg:** $\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| := \sqrt{x^2 + y^2 + z^2}$

**Sanity Check:** $c \in \mathbb{R} \quad v \in \mathbb{R}^n$

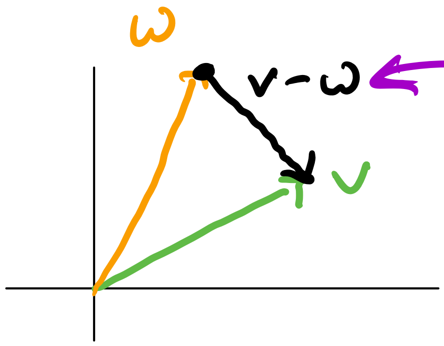
$$\|cv\| = \left\| c \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \right\| = \sqrt{(cx_1)^2 + \cdots + (cx_n)^2}$$

$$= |c| \cdot \sqrt{x_1^2 + \cdots + x_n^2} = |c| \cdot \|v\| \quad \checkmark$$

**Eg:** $2v$ is twice as long as $v$

So is $-2v$.

**Def:** The distance from $v$ to $w$ is $\|v-w\| = \|w-v\|$




length of $v-w$ is distance from $v$ to $w$

**Def:** A unit vector is a vector of length 1.

ie $\|v\| = 1$ ie. $\|v\|^2 = 1$

ie $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then $v$ is a unit vector

$\iff x_1^2 + \cdots + x_n^2 = 1$

$\iff v$ lies on the unit $(n-1)$-sphere

($n=2$: unit circle)

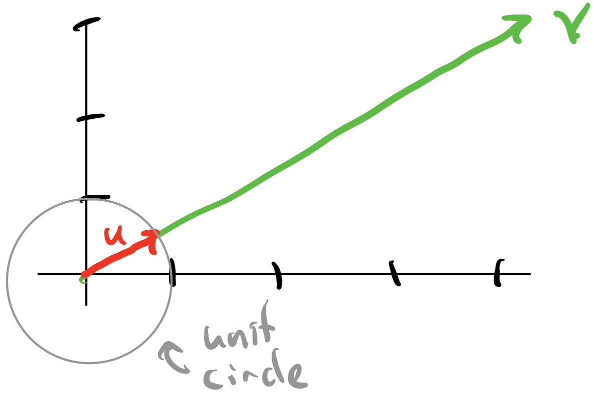
If $v \neq 0$, the unit vector in the direction of $v$ is the vector

$$u = \frac{1}{\|v\|} \cdot v = \frac{v}{\|v\|} \qquad \text{(scalar} \times \text{vector)}$$

**NB:** $\|u\| = \left|\frac{1}{\|v\|}\right| \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1$ ✓

**Eg:** $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\quad \|v\| = \sqrt{3^2+4^2} = 5$

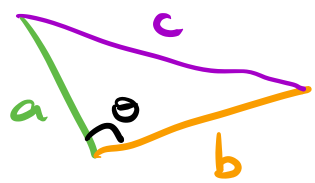
$$u = \frac{1}{\|v\|} v = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$




**NB:** all unit vectors in $\mathbb{R}^2$ are on the unit circle.

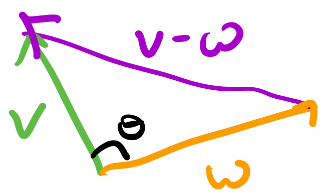
What about $v \cdot w$ for $v \neq w$?

## Law of Cosines:




$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

## Vector Version:




$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$$

$a = \|v\|$ $\quad b = \|w\|$ $\quad c = \|v-w\|$

"left hand side" LHS:

$$\|v-w\|^2 := (v-w)\cdot(v-w)$$

FOIL

$$= v\cdot v + w\cdot w - 2v\cdot w$$

$$= \cancel{\|v\|^2} + \cancel{\|w\|^2} \cancel{-2} v\cdot w$$

$$= \cancel{\|v\|^2} + \cancel{\|w\|^2} \cancel{-2}\|v\|\|w\|\cos\theta$$

$$\boxed{v\cdot w = \|v\|\|w\|\cos\theta \quad \text{or} \quad \cos\theta = \frac{v\cdot w}{\|v\|\|w\|} \quad (\text{if } v,w \neq 0)}$$

**Def:** The angle from $v$ to $w$ $(v, w \neq 0)$ is

$$\theta := \cos^{-1}\left(\frac{v \cdot w}{\|v\|\|w\|}\right)$$

**NB:** $|\cos\theta| = \left|\frac{v \cdot w}{\|v\|\|w\|}\right| \in [0, 1]$

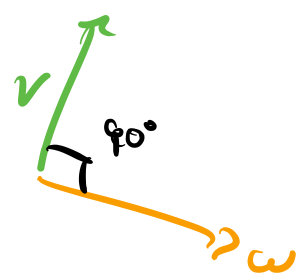
$$\Rightarrow \quad |v \cdot w| \leq \|v\| \cdot \|w\|$$

**Schwartz Inequality:** $|v \cdot w| \leq \|v\| \cdot \|w\|$ ✓

**Def:** Vectors $v$ and $w$ are **orthogonal** or **perpendicular** if $\boxed{v \cdot w = 0}$

This says that either:

- $v = 0$ or $w = 0$ (or both), or
- $\cos(\theta) = 0 \iff \theta = \pm 90°$



# Matrix Algebra

A **matrix** is a box holding a grid of numbers:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \qquad \cancel{\begin{matrix} 1 & 4 \\ 2 & 3 \\ 5 & 6 \end{matrix}}$$

$m = 3$ rows

$n = 2$ columns

$m \times n = 3 \times 2$ is the **size** of the matrix.

$\bullet = (3,2)$-**entry** : number in $3^{rd}$ row, $2^{nd}$ col.

**component**

**Diagonal entries** are the $(i,i)$-entries

$$\begin{bmatrix} 1 & 4 & 11 \\ 2 & 5 & -1 \\ 3 & 7 & -2 \\ 0 & 2 & \pi \end{bmatrix} \qquad \begin{matrix} (1,1) \\ (2,2) \\ (3,3) \end{matrix}$$

**Addition & Scalar Multiplication** are done componentwise (like for vectors):

$$c \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} cx_{11} & cx_{12} \\ cx_{21} & cx_{22} \\ cx_{31} & cx_{32} \end{bmatrix}$$

same size

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{bmatrix} = \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} \\ x_{21}+y_{21} & x_{22}+y_{22} \\ x_{31}+y_{31} & x_{32}+y_{32} \end{bmatrix}$$

NB: Only makes sense to add/subtract matrices of the same size.

NB: A vector is just a matrix with 1 column! (an $n \times 1$ matrix)

Def: A row vector is a matrix with one row (a $1 \times n$ matrix) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

Matrix × Vector: $\rightsquigarrow$ vector

get the same answer either way

(1) Column first:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} := c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

linear combination of the columns

!! The components of the vector $v$ are the weights of $Av$ = linear combination of the columns of $A$.

(2) Row first: (do it this way by hand):

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \cdot c_1 + 4 \cdot c_2 \\ 2 \cdot c_1 + 5 \cdot c_2 \\ 3 \cdot c_1 + 6 \cdot c_2 \end{bmatrix}$$

!! The $i^{th}$ entry of $Av$ is the dot product of the $i^{th}$ row of $A$ with the vector $v$.

NB: Only makes sense if #columns of $A$ equals the size of $v$.

$$A \text{ is } m \times n, \quad v \in \mathbb{R}^n \rightsquigarrow Av \in \mathbb{R}^m$$

$$(m \times \cancel{n}) \cdot (\cancel{n} \times 1) \rightsquigarrow (m \times 1)$$

Eg:
$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\overset{\text{1st col}}{=} -1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$\overset{\text{1st row}}{=} \begin{bmatrix} 1(-1) + 4(1) \\ 2(-1) + 5(1) \\ 3(-1) + 6(1) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

Eg:
$$\begin{bmatrix} 1 & 4 & 7 & 10 \\ 2 & 5 & 8 & 11 \\ 3 & 6 & 9 & 12 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\overset{\text{1st col}}{=} \cancel{0 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} + 1 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \cancel{0 \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}} + \cancel{0 \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$= 2^{nd}$ column

$\boxed{\text{Fact: } \quad Ae_i = i^{\text{th}} \text{ column of } A}$

Matrix × Matrix: $\rightsquigarrow$ matrix     3 ways:

yield the same answer

(1) Column first:

$$A\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ Au & Av & Aw \\ | & | & | \end{bmatrix}$$

"distributes over cols of $2^{\text{nd}}$ matrix"

(2) Rows first: (do it this way by hand):

$$\underset{2\times 4}{\begin{bmatrix} - & x & - \\ - & y & - \end{bmatrix}}\underset{4\times 3}{\begin{bmatrix} | & | & | \\ u & v & w \\ | & | & | \end{bmatrix}} = \begin{bmatrix} x\cdot u & x\cdot v & x\cdot w \\ y\cdot u & y\cdot v & y\cdot w \end{bmatrix} \quad \rightsquigarrow \quad 2\times 3$$

(3) postpone for a page

NB: $AB$ only makes sense if

\# columns of $A$ = \# rows of $B$.

$$(m\times \cancel{n})\cdot(\cancel{n}\times p) \rightsquigarrow m\times p$$

Eg:

$$\underset{2\times 3}{\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix}}\underset{3\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} \phantom{0} & \phantom{0} \\ \phantom{0} & \phantom{0} \end{bmatrix}}$$

(1) Column firsts

$$= \left[ \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 17 & 2 \\ -13 & 3 \end{bmatrix}$$

(2) Row first:

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 \cdot 3 \cdot 4 & 1\cdot 3 + 2\cdot 1 + 3(-1) \\ (-1)1 + 2\cdot 2 + (-4)4 & (-1)\cdot 3 + 2\cdot 1 + (-4)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 2 \\ -13 & 3 \end{bmatrix} \quad \checkmark$$

**Def:** A column vector × row vector is called an **outer product**.

$$\underset{(m\times 1)}{\begin{bmatrix} | \\ u \\ | \end{bmatrix}} \underset{(1\times n)}{\begin{bmatrix} - & v & - \end{bmatrix}} = \underset{m\times n}{\begin{bmatrix} \text{products of} \\ \text{pairs of} \\ \text{entries of} \\ u \,\&\, v \end{bmatrix}}$$

Eg:
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$$

$(3)^{rd}$ way: outer product form:

$$\underbrace{\begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix}}_{n \text{ cols}} \underbrace{\begin{bmatrix} — & v_1 & — \\ — & v_2 & — \\ — & v_3 & — \end{bmatrix}}_{n \text{ rows}}$$

$$= \begin{bmatrix} | \\ x_1 \\ | \end{bmatrix} \begin{bmatrix} — & v_1 & — \end{bmatrix} + \begin{bmatrix} | \\ x_2 \\ | \end{bmatrix} \begin{bmatrix} — & v_2 & — \end{bmatrix} + \begin{bmatrix} | \\ x_3 \\ | \end{bmatrix} \begin{bmatrix} — & v_3 & — \end{bmatrix}$$

Eg:
$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \end{bmatrix} \begin{bmatrix} 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 12 & -3 \\ -16 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 2 \\ -13 & 3 \end{bmatrix} \checkmark$$

**Def:** The $n \times n$ identity matrix is

$$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

$\nwarrow$ $i^{\text{th}}$ column is $e_i$

This is a square matrix (# rows $= n =$ # cols)

**Eg:** $$AI_n = A\begin{bmatrix} | & & | \\ e_1 & \cdots & e_n \\ | & & | \end{bmatrix} \overset{\substack{\text{cols} \\ 1^{\text{st}}}}{=} \begin{bmatrix} | & & | \\ Ae_1 & \cdots & Ae_n \\ | & & | \end{bmatrix} = A$$

($Ae_i = i^{\text{th}}$ col of $A$)

**Eg:** $I_m A = A$

**Eg:** The $m \times n$ zero matrix is $0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$

**Def:** The transpose of a matrix $A$ is the matrix $A^T$ whose columns are the rows of $A$.

$$A = \begin{bmatrix} - & x_1 & - \\ & \vdots & \\ - & x_m & - \end{bmatrix} \qquad A^T = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_m \\ | & & | \end{bmatrix}$$

$(m \times n)$ $\qquad$ $(n \times m)$

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & -4 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -4 \end{bmatrix}$

flip over diagonal

Eg: If $v, w \in \mathbb{R}^n$ $\quad v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad w = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$$v^T w = [x_1 \cdots x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = [x_1 y_1 + \cdots + x_n y_n]$$

$$= [v \cdot w]$$

$$\boxed{v^T w = v \cdot w}$$

(a $1 \times 1$ matrix is a scalar)

Compare: $v, w \in \mathbb{R}^n$

$v^T w$: inner (dot) product: $1 \times 1$

$v w^T$: outer product: $n \times n$

Def: If $A$ is a square matrix ($n \times n$) then

$$A^2 = A \cdot A \qquad A^3 = A \cdot A \cdot A \qquad A^4 = A \cdot A \cdot A \cdot A, \ldots$$

are the powers of $A$

Eg: $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$

NB: $A \cdot A$ only makes sense if $m = n$! $\quad (m \times n) \cdot (m \times n)$

Eg: What about $A^{-1}$? (Week 2.)

Rules for Matrix Algebra: ASSUME SIZES ARE COMPATIBLE

- Associativity: $(AB)C = A(BC)$

  $\rightsquigarrow$ can write $ABC$

  Eg: If $C=v$ is a vector, then

  $$A(Bv) = (AB)v$$

- Distributivity:

  $A(B+C) = AB+AC$ $\quad$ $(A+B)C = AC+BC$

- Identity: $A$ is $m \times n$

  $$I_m A = A = A I_n$$

- $(A^T)^T = A$ $\quad$ • $(A+B)^T = A^T + B^T$

- $(AB)^T = \cancel{A^T B^T}$ $\quad$ $B^T A^T$ ✓

  $A^T B^T$ may not be defined: $A: 2\times 3$, $B: 3\times 4$

  $AB$ $(2\times 3)\cdot(3\times 4) \rightsquigarrow (2\times 4)$

  $A^T B^T$ $(3\times 2)\cdot(4\times 3) \rightsquigarrow$ ✗

  $B^T A^T$ $(4\times 3)\cdot(3\times 2) \rightsquigarrow (4\times 2)$ ✓

## Commutativity fails!!

$$AB \neq BA$$

↳ $BA$ may not make sence

↳ or: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\neq$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

## Cancellation fails!!

$$AB = AC \text{ and } A \neq 0 \not\Rightarrow B = C$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = AC \quad B \neq C$$