Number of Solutions

The most basic question you can ask about a system of equations is: how many solutions does it have?

Recall: The pivot positions (pivots) of a matrix are the first nonzero entries of each row after putting the matrix into REF (using row operations).

The rank of a matrix is the number of pivots

Eg: from last time:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -\tfrac{50}{7}x_3 &= -\tfrac{150}{7} \end{aligned}$$

How many solutions? 1

Find the (only) solution using back-substitution.

Eg: from last time:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} x_1+2x_2+3x_3&=1\\ 4x_1+5x_2+6x_3&=0\\ 7x_1+8x_2+9x_3&=-1 \end{aligned} \quad \hookrightarrow \quad \begin{aligned} x_1+2x_2+3x_3&=1\\ -3x_2+6x_3&=4\\ 0&=0 \end{aligned}$$

How many solutions? ∞

We can choose any value for x_3: no pivot in 3^{rd} col.

Q: What is the difference between the previous two examples? (In terms of pivots.)

Eg: from last time:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 4 & 5 & 6 & 0\\ 7 & 8 & 9 & 0 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & -3 & -6 & 4\\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{aligned} x_1+2x_2+3x_3&=1\\ 4x_1+5x_2+6x_3&=0\\ 7x_1+8x_2+9x_3&=-1 \end{aligned} \quad \hookrightarrow \quad \begin{aligned} x_1+2x_2+3x_3&=1\\ -3x_2+6x_3&=4\\ 0&=1 \end{aligned}$$

How many solutions? 0

Q: What is the difference between the previous three examples? (In terms of pivots.)

Def: A pivot column of a matrix is a column with a pivot position.

$\rightarrow$ Again, the pivots are in the REF matrix!

Fact: Let A be the augmented matrix for a system of equations.

(1) If every column except the last is a pivot column, then the system has one solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

= pivot column

= not a pivot column

(∞) If the last column and some other column are not pivot columns, then there are infinitely many solutions.

can choose any value for x_3

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

= pivot column

= not a pivot column

(0) If the last column is a pivot column, then there are zero solutions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

= pivot column

= not a pivot column

Def: A system is consistent if it has a solution. It is inconsistent otherwise. (third case above)

Gaussian Elimination

This is how a computer solves systems of linear equations using elimination. Almost all questions in this class will reduce to this procedure! (The interesting part is how they do so.)

Def: Two matrices are **row equivalent** if you can get from one to the other using row operations.

NB: If augmented matrices are row equivalent then they have the same solution sets.

Algorithm (Gaussian Elimination/row reduction):

Input: Any matrix

Output: A row-equivalent matrix in REF.

Procedure:

(1a) If the first nonzero column has a zero entry at the top row swap so that the top entry is nonzero.

$$\begin{bmatrix} 0 & 4 & 3 & 2 \\ 1 & 1 & -1 & 3 \\ 2 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 2 & -3 & -6 & -3 \end{bmatrix}$$

This is now the first pivot position.

(1b) Perform row replacements to clear all entries below the first pivot.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 2 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{R_3 -= 2R_1} \begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 0 & -5 & -4 & -9 \end{bmatrix}$$

Now ignore the row & column with the first pivot and recurse into the submatrix below and to the right:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 0 & -5 & -4 & -9 \end{bmatrix}$$

(2a) If the first nonzero column has a zero entry at the top row swap so that the top entry is nonzero.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 0 & -5 & -4 & -9 \end{bmatrix}$$

second pivot

(Not applicable to this matrix)

(2b) Perform row replacements to clear all entries below the first pivot.

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 0 & -5 & -4 & -9 \end{bmatrix} \xrightarrow{R_3 += \frac{5}{4}R_2} \begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1/4 & -13/2 \end{bmatrix}$$

Doesn't mess up the 1st column!

etc. (recurse)

In our example, the recursion has terminated:

$$\begin{bmatrix} 1 & 1 & -1 & -3 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1/4 & -13/2 \end{bmatrix}$$ is in REF!

Eg:
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{(1b)}]{R_2 -= 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

This submatrix has no pivot in the first column! The first nonzero column is the second.

$$\xrightarrow[\text{(2b)}]{R_3 -= R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

This is in REF: $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Important: If you want to apply this algorithm to an augmented matrix, just delete the augmentation line (pretend it's not augmented).

Demo: Gauss-Jordan slideshow

Use Rabinoff's Reliable Row Reducer on the HW!

Jordan Substitution

This is the back-substitution procedure.

It is necessary when you have ∞ solutions.

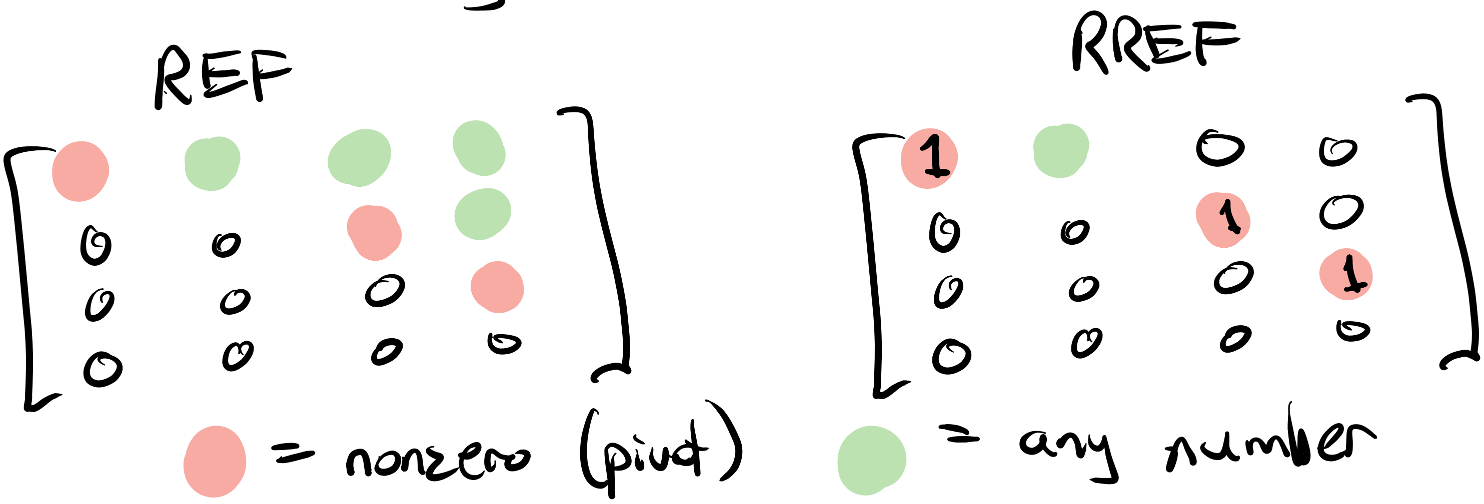
It puts a matrix into the following form:

Def: A matrix is in reduced row echelon form (RREF) if:

(1–2) It is in REF

(3) All pivots are equal to 1.

(4) A pivot is the only nonzero entry in its column.



Eg: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$ is in REF. How to put into RREF?

Do back substitution!

Row Operations | **Back-Substitution**

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -\tfrac{50}{7}x_3 &= -\tfrac{150}{7} \end{aligned}$$

(scale so this is 1)

$R_3 \times = -\frac{7}{50}$ — solve for x_3

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ x_3 &= 3 \end{aligned}$$

(kill these)

$R_1 -= 3R_3$
$R_2 += 4R_3$
— substitute $x_3 = 3$ into R_1 & R_2 then move the constants to the RHS

$$\left[\begin{array}{cccc} 1 & 2 & 0 & -3 \\ 0 & -7 & 0 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 &= -3 \\ -7x_2 &= 14 \\ x_3 &= 3 \end{aligned}$$

(scale so this is 1)

$R_2 \div= -7$ — solve for x_2

$$\left[\begin{array}{cccc} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 2x_2 &= -3 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

(kill this)

$R_1 -= 2R_2$ substitute $x_2 = -2$ into R_1 then move the constants to the RHS

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

This is in RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightsquigarrow \begin{aligned} x_1 &= 1 \\ x_2 &= -2 \\ x_3 &= 3 \end{aligned}$$

Solved ✓

Upshot: Jordan substitution is exactly back-substitution.

Demo: Gauss-Jordan slideshow, cont'd

Algorithm (Jordan Substitution):

Input: A matrix in REF

Output: The row-equivalent matrix in RREF.

Procedure:

Loop, starting at the last pivot:

(a) Scale the pivot row so the pivot $=1$.

(b) Use row replacements to kill the entries above that pivot.

"theorem"

Thm: The RREF of a matrix is unique.

In other words, if you start with a matrix, do any legal row operations at all, and end with a matrix in RREF, then it's the same matrix that Gauss-Jordan will produce.

↳ Gaussian elimination + Jordan substitution.

Inverse Matrices

This is an application of Gauss-Jordan.

It answers the question: to solve $Ax=b$, can we divide by A?

(this is the coefficient matrix not the augmented matrix)

Def: An $n\times n$ (square!) matrix A is invertible if there exists another $n\times n$ matrix B such that
$$AB = I_n = BA.$$

$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $n\times n$ identity matrix

A matrix with an inverse is called invertible. Otherwise it's called singular.

Notn: $B = A^{-1}$, called the inverse of A.

Eg: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow B = A^{-1}$$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{\text{“B”}} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq I_2$

so A is singular (non-invertible).