Parametric Form

Now we deal systematically with systems of equations with ∞ solutions. We want to parameterize all solutions.

Eg:
$$\begin{aligned} 2x+y+12z&=1 \\ x+2y+9z&=-1 \end{aligned} \rightsquigarrow \left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \rightsquigarrow \begin{aligned} x+5z&=1 \\ y+2z&=-1 \end{aligned}$$

Observation: If you substitute any number for z, you get the system

unknowns

$$\left[\begin{aligned} x &= 1-5z \\ y &= -1-2z \end{aligned}\right.$$

numbers

which has a unique solution!

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$$

eg $z=1$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$

check:
$$\begin{aligned} 2(-4)+(-3)+12(1)&=1 \\ -4+2(-3)+9(1)&=-1 \ \checkmark \end{aligned}$$

This is the parametric form of the solution; z is the free variable or parameter.

Implicit vs Parameterized Form.

The system of equations

$$\begin{cases} 2x+y+12z=1 \\ x+2y+9z=-1 \end{cases}$$

are implicit equations of a line: it expresses the line as the set of solutions of these equations without giving you any way to write down specific points on the line. The parametric form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-5z \\ -1-2z \\ z \end{pmatrix}$$

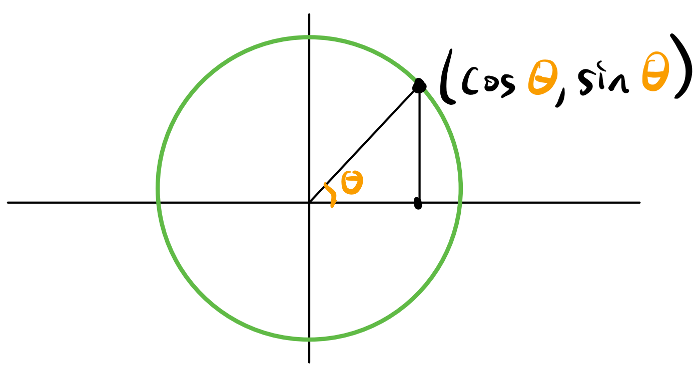
is a parametric equation for the same line: it gives you a way to produce all solutions in terms of the parameter z.

[demo]

Non-linear example:

An implicit equation for the unit circle is

$$x^2+y^2=1$$



A parametric equation for the unit circle is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad \theta = \text{parameter}$$

Here's how to produce parametric equations for general linear systems.

Recall: A pivot column of a matrix is a column with a pivot.

Def: A free variable in a system of equations is a variable whose column (in the coeff matrix) is not a pivot column.

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \\ \begin{array}{ccc} x & \;\; y & \;\; z \end{array} \end{array}$$

x, y in pivot cols

z is free

These are the variables you can't solve for in back-substitution.

Procedure (Parametric Form):

To find the parametric form of the solutions of $Ax=b$:

(1) Put $[A \mid b]$ into RREF. Stop if inconsistent.

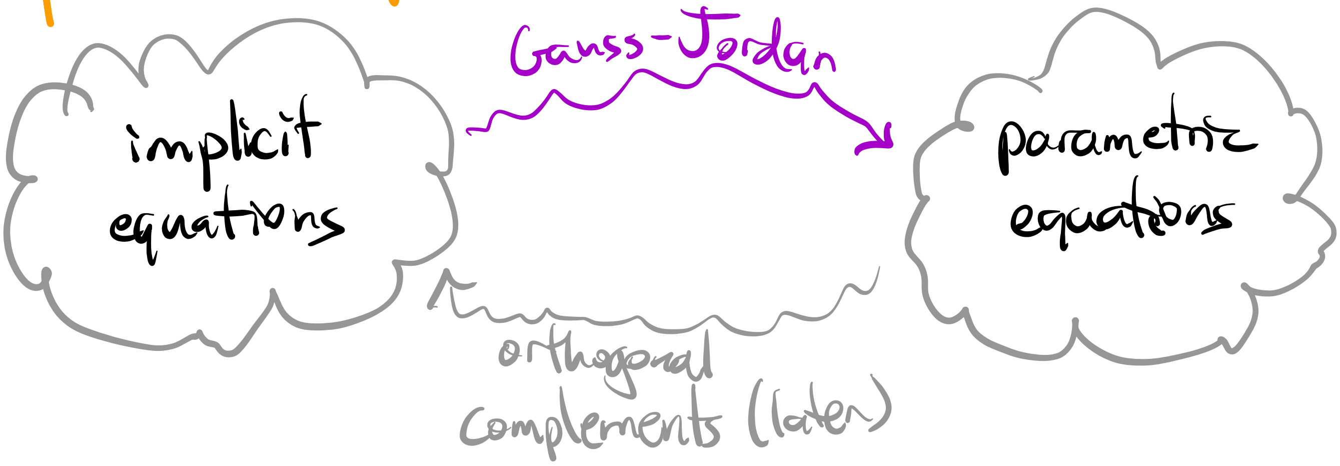
(2) Write out the corresponding equations

(3) Move free variables to the right-hand side

All solutions are obtained by substituting any values for the free variables.

This uses the free variables as the parameters.

So Gauss-Jordan elimination turns implicit equations (step (1) above) into parametric equations.



Eg: $x+y+z=1 \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right]$

This is already in RREF!

Free variables: y, z

$$\xrightarrow[\text{form}]{\text{parametric}} \quad x = 1-y-z$$

This is a parameterized plane. [demo]

Eg: $\begin{aligned} x+y&=2 \\ x-y&=0 \end{aligned} \rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

No free variables! Just have one solution $x=1, y=1$.

Observation:

- 2 free variables / 2 parameters:
  solution set is a plane
- 1 free variable / 1 parameter:
  solution set is a line
- 0 free variables / 0 parameters:
  solution set is a point

Provisional Def^n: The dimension of the solution set of a consistent system $Ax=b$ is the number of free variables.

Parametric Vector Form

This is an alternate, more geometric way of writing a solution set in parametric form.

Eg: $\begin{aligned} 2x+y+12z&=1 \\ x+2y+9z&=-1 \end{aligned} \xrightarrow[\text{form}]{\text{parametric}} \underbrace{\begin{aligned} x&=1-5z \\ y&=-1-2z \\ z&=z \end{aligned}}_{\text{parameterize the free variable too}}$ (from before)

Let's rewrite this as an equation involving vectors:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

This is the line thru $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in the $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$-direction.

[demo again]

note nice columns

Eg: $x+y+z=1 \xrightarrow[\text{form}]{\text{parametric}} \begin{aligned} x&=1-y-z \\ y&=y \\ z&=z \end{aligned}$ } parameterize the free variables

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This is the plane containing $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, & $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}+\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

[demo again]

Writing the solution set in this way is called the parametric vector form.

Procedure (Parametric Vector Form)

To find the parametric vector form of the solutions of $Ax=b$:

(1–3) Find the parametric form

(4) Add trivial equations for the free variables, in order. Organize the right-hand side into columns.

(5) Gather the columns into vectors. Pull out the free variables as coefficients.

Result:

$$x = \begin{pmatrix} \text{a constant} \\ \text{vector} \end{pmatrix} + \begin{pmatrix} \text{a linear combination with} \\ \text{the free variables as coefficients} \end{pmatrix}$$

NB: The constant vector is the solution you get by setting all free variables $=0$.

Def: This vector is called the particular solution. (It is a solution of $Ax=b$)

Eg: $\begin{aligned} x+2y+2z+w&=1 \\ 2x+4y+z-w&=-1 \end{aligned} \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & 1 & -1 & -1 \end{array}\right]$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

$$\rightsquigarrow \begin{aligned} x+2y \quad -w &= -1 \\ z+w &= 1 \end{aligned}$$

free

$$\rightsquigarrow \begin{cases} x = -1 - 2y + w \\ y = y \\ z = 1 - w \\ w = w \end{cases}$$

trivial equations

columns

$$\xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

particular solution

any linear combination of $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Vector Equations

This is another way of writing a linear system that is useful for geometric reasoning.

Def: A vector equation is an equation involving linear combinations of vectors with unknown coefficients.

Eg: $x_1\begin{pmatrix}1\\2\\6\end{pmatrix}+x_2\begin{pmatrix}-1\\-2\\-1\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix}$

This is equivalent to the system

$$\begin{pmatrix}1&-1\\2&-2\\6&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}8\\16\\3\end{pmatrix}$$

(use the column-first definition of the matrix-vector product). But now we're thinking geometrically about linear combinations of vectors.

Four Ways to Write a System of Eqns:

(1) Linear system

$$\begin{aligned} x_1 - x_2 &= 8 \\ 2x_1 - 2x_2 &= 16 \\ 6x_1 - x_2 &= 3 \end{aligned}$$

(2) Matrix Equation

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(columns)

(3) Augmented Matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right)$$

(4) Vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

You still solve a vector equation by putting it into an augmented matrix:

Eg: $$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right]$$

Solution is $x_1 = -1$, $x_2 = -9$

Important Observation: (!!!!)

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$ has a solution (consistent)

$\iff$ $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ is a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$

in which case the solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of coefficients.

In fact, we know $x_1 = -1$, $x_2 = -9$:

$$-1\begin{pmatrix}1\\2\\6\end{pmatrix} - 9\begin{pmatrix}-1\\-2\\-1\end{pmatrix} = \begin{pmatrix}8\\16\\3\end{pmatrix} = \left(\begin{matrix}\text{linear combination}\\ \text{of } \begin{pmatrix}1\\2\\6\end{pmatrix} \ \& \ \begin{pmatrix}-1\\-2\\-1\end{pmatrix}\end{matrix}\right)$$

[demo: consistent]

different b

Eg: $$x_1\begin{pmatrix}1\\2\\6\end{pmatrix} + x_2\begin{pmatrix}-1\\-2\\-1\end{pmatrix} = \begin{pmatrix}2\\-2\\0\end{pmatrix} \rightsquigarrow \left[\begin{array}{cc|c}1 & -1 & 2\\2 & -2 & -2\\6 & -1 & 0\end{array}\right]$$

$$\overset{\text{REF}}{\rightsquigarrow} \left[\begin{array}{cc|c}1 & -1 & 2\\0 & 5 & -12\\0 & 0 & -6\end{array}\right]$$ inconsistent

So $\begin{pmatrix}2\\-2\\0\end{pmatrix}$ is not a linear combination of $\begin{pmatrix}1\\2\\6\end{pmatrix}$ & $\begin{pmatrix}-1\\-2\\-1\end{pmatrix}$.

[demo: inconsistent]

Now we have two pictures of when a system is consistent/inconsistent:

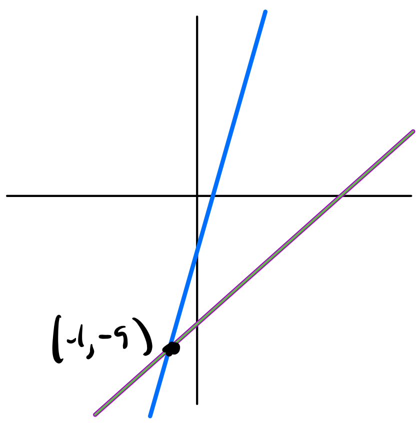
- **Row Picture:**

  Draw pictures of the solution set of each equation (row). Consistent means there's a vector in the intersection.

  n variables: picture in $\mathbb{R}^n$

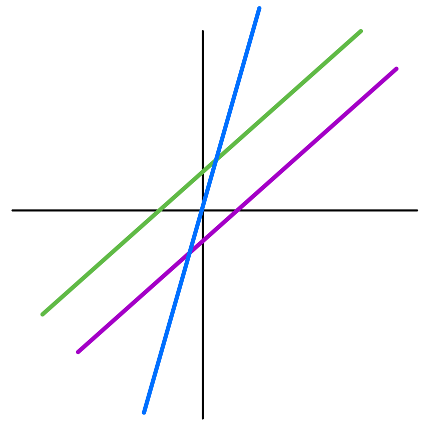
consistent

$x_1 - x_2 = 8$
$2x_1 - 2x_2 = 16$
$6x_1 - x_2 = 3$



inconsistent

$x_1 - x_2 = 2$
$2x_1 - 2x_2 = -2$
$6x_1 - x_2 = 0$



- **Column Picture:**

  Draw all possible linear combinations of the columns of the coefficient matrix. Consistent means b is one of these points.

  m equations: picture in $\mathbb{R}^m$

  [look at demos again]

NB: parametric vector form lives in the row picture: you're drawing solutions in $\mathbb{R}^n$.