Spans

Recall from last time:

(1) Parametric Vector Form: The solutions of $Ax=b$ can be written as

$$x = \begin{pmatrix}\text{constant} \\ \text{vector}\end{pmatrix} + \begin{pmatrix}\text{any LC of some vectors} \\ \text{with coeffs = free variables}\end{pmatrix}$$

the particular solution

eg $\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix} x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\rightsquigarrow x = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

(2) Column picture: $Ax=b$ has a solution (consistent)
$\iff$ b is a linear combination of the columns of A.

eg. $$\begin{bmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}}^{\text{LC of columns}} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$$

Both involve understanding all linear combinations of a set of vectors. This has a name.

Def: The span of a list of vectors is the set of all linear combinations of those vectors:

$$\text{Span}\{v_1, v_2, \ldots, v_n\} = \underbrace{\{}_{\text{"the set of"}} \underbrace{c_1v_1 + c_2v_2 + \cdots + c_nv_n}_{\text{"all things of this form"}} \underbrace{:}_{\text{"such that"}} \underbrace{c_1, \ldots, c_n \in \mathbb{R}}_{\text{"these conditions hold"}} \}$$

This is set-builder notation

Translation of p.1:

(1) The solution set of $Ax=b$ is $\begin{pmatrix}\text{particular}\\ \text{solution}\end{pmatrix} + \text{Span}\left\{\begin{matrix}\text{some}\\ \text{vectors}\end{matrix}\right\}$

$$\boxed{(2)\ Ax=b \text{ is consistent} \iff b \in \text{Span}\{\text{cols of } A\}}$$

!!!

What do spans look like?

It's the smallest "linear space" (line, plane, etc.) containing all your vectors & the origin.

Eg: $\text{Span}\{v\} = \{cv : c \in \mathbb{R}\}$

$\to$ If $v \neq 0$ get the line thru 0 & v

$\to$ $\text{Span}\{0\} = \{c \cdot 0 : c \in \mathbb{R}\} = \{0\}$

= the set containing only 0 [demo]

Eg: $\mathrm{Span}\{v,w\} = \{cv+dw : c,d \in \mathbb{R}\}$

→ If v,w are not collinear, get the plane defined by 0, v, and w

→ If v,w are collinear and nonzero, get the line thru v,w, and 0.

→ If $v=w=0$ get $\{0\}$ [demo]

Eg: $\mathrm{Span}\{u,v,w\} = \{bu+cv+dw : b,c,d \in \mathbb{R}\}$

→ If u,v,w are not coplanar, get space

→ If u,v,w are coplanar but not collinear, get the plane containing them.

→ If u,v,w are collinear & not all zero, get the line thru u,v,w, and 0.

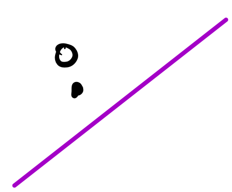
→ If $u=v=w=0$ get $\{0\}$ [demo]

Eg: $\mathrm{Span}\{\} = \{0\}$ (by convention)

$$\boxed{\text{NB: Every span contains the zero vector!}}$$

$$0 = 0\cdot v_1 + 0\cdot v_2 + \cdots + 0\cdot v_n$$

So eg. this line is not a span: →



Eg: $\{\}$ is not a span! It does not contain 0.

Eg: Is $\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$ in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} \right\}$?

In other words, does

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$$ have a solution?

Let's solve this vector equation:

$$\left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} x_1 = -1 \\ x_2 = -9 \end{array}$$

[demo]

Answer: yes

This example is just the "$\Leftarrow$" of the statement:

$$\boxed{(2)\ Ax = b \text{ is consistent} \iff b \in \text{Span}\{\text{cols of } A\}}$$

(copied from p.2 for emphasis)

Homogeneous Equations

If the solution set of $Ax=b$ is a span

$\Rightarrow 0$ is a solution (every span contains 0)

$\Rightarrow A0=b \Rightarrow b=0$

Let's study this case.

Def: $Ax=b$ is called **homogeneous** if $b=0$.

Eg:
$$x_1+2x_2+2x_3+x_4=0$$
$$2x_1+4x_2+x_3-x_4=0$$

NB: A homogeneous equation is always consistent since 0 is a solution: $A\cdot 0=0$

Def: The **trivial solution** of a homogeneous equation $Ax=0$ is the zero vector.

Eg: Let's solve the homogeneous system

$$x_1+2x_2+2x_3+x_4=0$$
$$2x_1+4x_2+x_3-x_4=0$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 -= 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 \div= -3} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 -= 2R_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

PF $\rightsquigarrow$

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= - x_4 \\ x_4 &= x_4 \end{aligned}$$

PVF $\rightsquigarrow$

$$x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Observations:

(1) The augmented column is always zero. When solving homogeneous equations, you don't need to write the augmented column.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array}\right] \rightsquigarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}$$

(2) The particular solution is the zero vector

(3) The solution set is

$$\text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

Fact: The PVF of a homogeneous system always has particular solution $= 0$. The solution set is the span of the other vectors you've produced.

Inhomogeneous Equations

Def: $Ax=b$ is called inhomogeneous if $\boxed{b\neq 0}$.

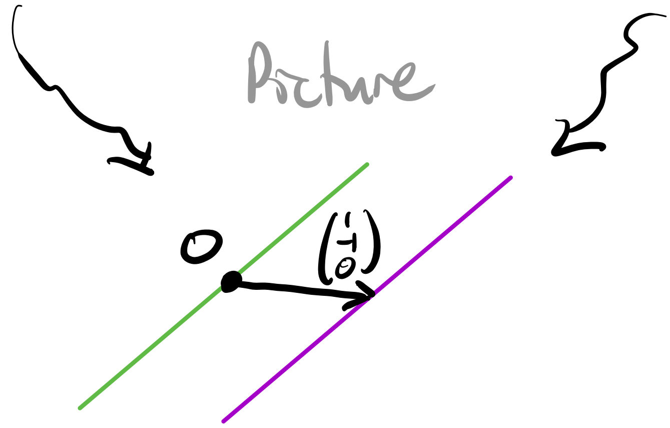
What's the difference from homogeneous equations?

NB: It can be inconsistent!

Let's solve the inhomogeneous & homogeneous versions:

Eg:

inhomogeneous		homogeneous
$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$		$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$\downarrow$	(augmented) matrix	$\downarrow$
$\left[\begin{array}{ccc\|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array}\right]$		$\left[\begin{array}{ccc\|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & 0 \end{array}\right]$
$\downarrow$	RREF	$\downarrow$
$\left[\begin{array}{ccc\|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$		$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$
$\downarrow$	PVF	$\downarrow$
$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$		$x = z\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$
$\downarrow$	Solution set	$\downarrow$
$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$		$\mathrm{Span}\left\{\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}\right\}$



The only difference is the particular solution!
Otherwise they're parallel lines. [demo]

Facts:

(1) The solution set of $Ax=0$ is a span.

(2) The solution set of $Ax=b$ is not a span for $b\neq 0$: it is a translate of the solution set of $Ax=0$ by a particular solution. (Or it is empty.)

In fact, to get the solutions of $Ax=b$ you can translate the solutions of $Ax=0$ by any single solution of $Ax=b$.

→ Say p is some solution of $Ax=b$, so $Ap=b$.
Then $Ax=0 \iff Ap+Ax=b \iff A(p+x)=b$

vectors of the form $p+$ (soln of $Ax=0$)

NB: Expressing a solution set as a (translate of a) span means writing it in parametric form:

$$x \in \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$\Longleftrightarrow \quad x = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

parameters

So think:

Spans $\equiv$ Parametric form

Row & Column Picture

We now know:

Row Picture

(1) (The solutions of $Ax=b$)

Span

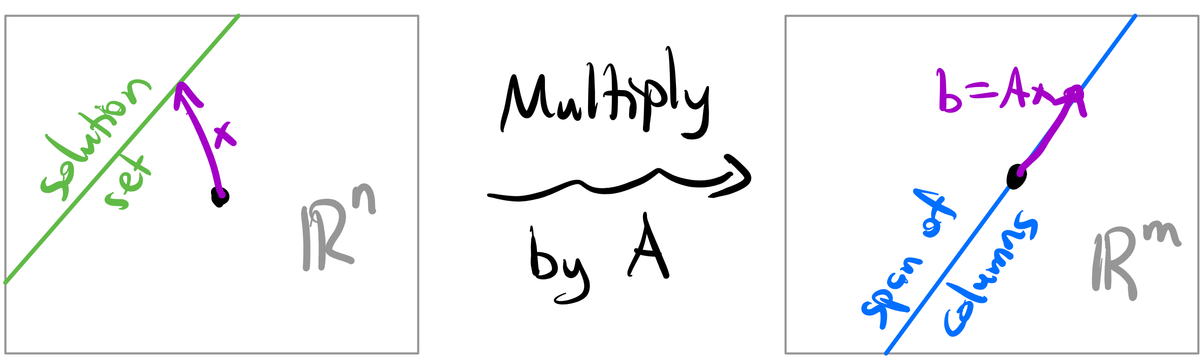
$= \left(\text{Some solution of } Ax=b\right) + \left(\text{all solutions of } Ax=0\right)$

or is empty. In particular, all nonempty solution sets are parallel.

Column Picture

(2) $Ax=b$ is consistent $\iff$ b is in the span of the columns of A.

We can draw these both at the same time:



[demo]

Many of the concepts on the first midterm appear in this picture.