Subspaces

So far, to every matrix A we have associated two spans:

(1) the span of the columns / all b such that $Ax=b$ is consistent

(2) the solution set of $Ax=0$

The first arises naturally as a span / it is already in parametric form. The second required Work (elimination) to write as a span — it is a solution set, so it is in implicit form.

The notion of subspaces puts both on the same footing. This formalizes what we mean by "linear space containing 0".

Fast-forward:

same picture

Subspaces are spans	and	Spans are subspaces.

Why the new vocabulary word?

When you say "span" you have a spanning set of vectors in mind (parametric form). This is not the case for the solutions of $Ax=0$.

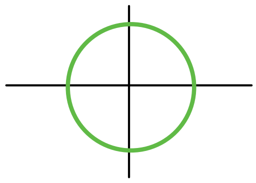
Subspaces allow us to discuss spans without computing a spanning set.

They also give a criterion for a subset to be a span.

Def: A subset of $\mathbb{R}^n$ is any collection of points.

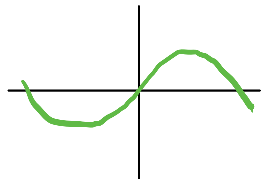
Eg:

(a)



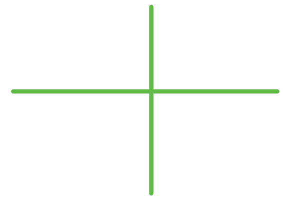
$\{(x,y) : x^2+y^2=1\}$

(b)



$\{(x,\sin(x)) : x\in\mathbb{R}\}$

(c)



$\{(x,y) : xy=0\}$

Def: A subspace is a subset V of $\mathbb{R}^n$ satisfying:

(1) [closed under $+$] If $u,v\in V$ then $u+v\in V$

(2) [closed under scalar $\times$]

If $u\in V$ and $c\in\mathbb{R}$ then $cu\in V$

(3) [contains 0] $0\in V$

These conditions characterize linear spaces containing 0 among all subsets.

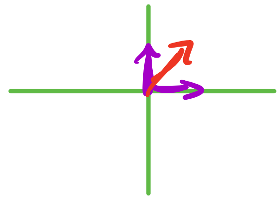
NB: If V is a subspace and $v\in V$ then $0=0v$ is in V by (2), so (3) just means V is nonempty

Eg: In the subsets above:

(a) fails (1), (2), (3)

(b) fails (1), (2)

(c) fails (1): $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \in V$ but $\begin{pmatrix}1\\1\end{pmatrix} \notin V$



Here are two "trivial" examples of subspaces:

Eg: $\{0\}$ is a subspace

(1) $0+0=0 \in \{0\}$ ✓

(2) $c \cdot 0 = 0 \in \{0\}$ ✓

(3) $0 \in \{0\}$ ✓

NB $\{0\} = \mathrm{Span}\{\}$: it is a span

Eg: $\mathbb{R}^n = \{\text{all vectors of size } n\}$ is a subspace

(1) The sum of two vectors is a vector. ✓

(2) A scalar times a vector is a vector. ✓

(3) 0 is a vector. ✓

NB $\mathbb{R}^n = \mathrm{Span}\{e_1, e_2, \ldots, e_n\}$

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

defining condition

Eg: $V = \{(x,y,z) : x+y=z\}$

The defining condition tells you if (x,y,z) is in V or not.

(1) We have to show that if $(x_1,y_1,z_1) \in V$ and $(x_2,y_2,z_2) \in V$ then their sum is in V. That means it also satisfies the defining condition.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix}$$

Is $z_1+z_2 = (x_1+x_2)+(y_1+y_2)$? Yes, because (x_1,y_1,z_1) and (x_2,y_2,z_2) satisfy the defining condition: $x_1+y_1=z_1$ $\quad x_2+y_2=z_2$

(2) We have to show that if $(x,y,z) \in V$ and $c \in \mathbb{R}$ then $c(x,y,z) \in V$.

$$c\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} \quad \text{is} \quad cx+cy=cz?$$

Yes, because $x+y=z$.

(3) Is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$? Does it satisfy the defining condition?

$$0+0=0$$

Since V satisfies the 3 criteria, it is a subspace.

Eg: $V = \{(x,y) : \overbrace{x \geq 0, y \geq 0}^{\text{defining condition}}\}$

(1) We have to show that if $(x_1,y_1) \in V$ and $(x_2,y_2) \in V$ then $(x_1+x_2, y_1+y_2) \in V$.

Is $x_1+x_2 \geq 0$? Yes, because $x_1 \geq 0$, $x_2 \geq 0$

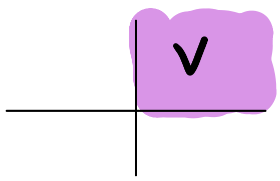
Is $y_1+y_2 \geq 0$? Yes, because $y_1 \geq 0$, $y_2 \geq 0$.

(3) Is $(0,0) \in V$? Yes: $0 \geq 0$ and $0 \geq 0$.

(2) We have to show that if $(x,y) \in V$ and $c \in \mathbb{R}$ then $(cx, cy) \in V$.

Is $cx \geq 0$? Not necessarily!

Fails if $c < 0$, $x > 0$.



Good: this is not a picture of a span.

In practice you will rarely check that a subset is a subspace by verifying the axioms.

Fact: A span is a subspace

Proof: Let $V = \text{Span}\{v_1, \ldots, v_n\}$.

Here the defining condition for a vec~~ta~~ to be in V is that it is a linear combination of $v_1, \ldots, v_n$.

(1) We need to show that if

$$c_1v_1 + \cdots + c_nv_n \in V \quad \& \quad d_1v_1 + \cdots + d_nv_n \in V$$

then their sum is in V: the sum of two linear combos of $v_1, \ldots, v_n$ is a linear combo.

$$(c_1v_1 + \cdots + c_nv_n) + (d_1v_1 + \cdots + d_nv_n)$$
$$= (c_1+d_1)v_1 + \cdots + (c_n+d_n)v_n \in V \quad \checkmark$$

(2) We need to show that if $c_1v_1 + \cdots + c_nv_n \in V$ and $d \in \mathbb{R}$ then the product is in V.

$$d(c_1v_1 + \cdots + c_nv_n) = (dc_1)v_1 + \cdots + (dc_n)v_n \in V \quad \checkmark$$

(3) Every span contains 0:

$$0 = 0v_1 + \cdots + 0v_n \quad \checkmark$$

Conversely, suppose V is a subspace.

If $v_1, \ldots, v_n \in V$ and $c_1, \ldots, c_n \in \mathbb{R}$ then:

$c_1v_1, \ldots, c_nv_n \in V$ by (2)

$c_1v_1 + c_2v_2 \in V$ by (1)

$(c_1v_1 + c_2v_2) + c_3v_3 \in V$ by (1)

$\vdots$

$c_1v_1 + \cdots + c_nv_n \in V$

So $\text{Span}\{v_1, \ldots, v_n\}$ is contained in V.

Choose enough v_i's to fill up V, and you get:

$\boxed{\text{Subspaces are spans} \quad \text{and} \quad \text{Spans are subspaces.}}$

Def: The column space of a matrix A is the span of its columns.

Notation: $\mathrm{Col}(A)$

This is a subspace of $\mathbb{R}^m$ $\quad m = \#\text{rows}$
(each column has m entries)

$\rightsquigarrow$ column picture.

Since a column space is a span & a span is a subspace, a column space is a subspace.

Eg: $\mathrm{Col}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$

It's easy to translate between spans & column spaces.

Eg: $\mathrm{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \mathrm{Col}\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

NB: $\boxed{\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^n\}}$

because "Ax" is just a LC of the cols of A.

Translation of the super-important fact from before:

$$\boxed{Ax=b \text{ is consistent} \iff b\in \mathrm{Col}(A)}$$

(this is just substituting "Col(A)" for "the span of the columns of A")

Def: The null space of a matrix A is the solution set of $Ax=0$.

Notation: $\mathrm{Nul}(A)$

This is a subspace of $\mathbb{R}^n$ $n=$ # columns
($n=$ # variables and $\mathrm{Nul}(A)$ is a solution set)

$\rightsquigarrow$ row picture

Fact: $\mathrm{Nul}(A)$ is a subspace

Of course we also know $\mathrm{Nul}(A)$ is a span, but we can verify this directly.

Proof: The defining condition for $v\in \mathrm{Nul}(A)$ is that $Av=0$.

(1) Say $u,v\in \mathrm{Nul}(A)$. Is $u+v\in\mathrm{Nul}(A)$?

$$A(u+v)=Au+Av=0+0=0 \quad ✓$$

(2) Say $u \in \text{Nul}(A)$ and $c \in \mathbb{R}$.

Is $cu \in \text{Nul}(A)$?

$$A(cu) = c(Au) = c \cdot 0 = 0 \quad ✓$$

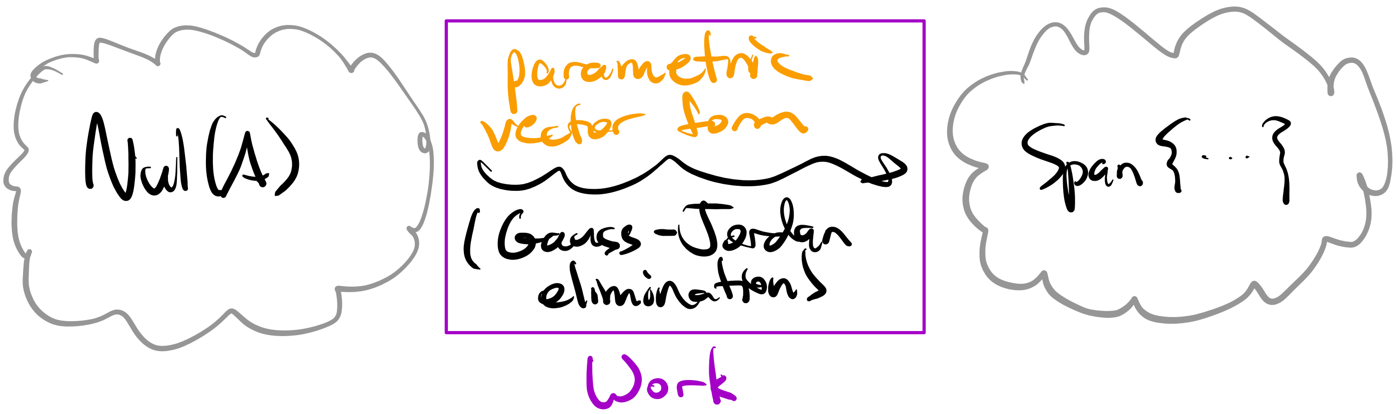
(3) Is $0 \in \text{Nul}(A)$?

$$A0 = 0 \quad ✓$$

This is an example of a subspace that does not come with a spanning set!

⇝ It's much more natural to consider it as a subspace when reasoning about it.

How to produce a spanning set for a null space?



Eg: Write Nul(A) as a span for

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}$$

This means solving $Ax = 0$ (homogeneous equation).

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{form}]{\text{parametric}} \begin{cases} x_1 = -2x_2 + x_4 \\ x_2 = \quad x_2 \\ x_3 = \quad -x_4 \\ x_4 = \quad x_4 \end{cases}$$

$$\xrightarrow{\text{PVF}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

NB: Any two non-collinear vectors span a plane, so Nul(A) will have many different spanning sets.

eg $\mathrm{Nul}(A) = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}}_{\text{sum}}, \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \\ -1 \end{pmatrix}}_{\text{difference}} \right\}$

More on this later.

Implicit vs Parametric form:

- Col (A) is a span:

$$\mathrm{Col}(A) = \{ x_1 v_1 + \cdots + x_n v_n : \underbrace{x_1, \ldots, x_n}_{\text{parameters}} \in \mathbb{R} \}$$

where $v_1, \ldots, v_n$ are the columns of A.

⇝ parametric form

- Nul (A) is a solution set:

$$\mathrm{Nul}\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}$$

$$= \left\{ (x_1, x_2, x_3, x_4) : \begin{array}{l} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 4x_2 + x_3 - x_4 = 0 \end{array} \right\}$$

⇝ implicit form

In practice you will (almost) always write a subspace as a column space/span or a null space. Which one?

- parameters? ⇝ Col (A) / Span
- equations? ⇝ Nul (A)

Once you're done this, you can ask a computer to do computations on it!

Eg: $V = \{(x,y,z) : x+y=z\}$

This is defined by the equation $x+y=z$.

rewrite: $x+y-z=0$

$\rightsquigarrow V = \text{Nul}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$

Eg: $V = \left\{ \begin{pmatrix} 3a+b \\ a-b \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

This is described by parameters. Rewrite:

$$\begin{pmatrix} 3a+b \\ a-b \\ b \end{pmatrix} = a\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow V = \text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = \text{Col}\begin{bmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

This is also how you should verify that a subset is a subspace.

(Of course, if V is not a subspace then you can't write it as $\text{Col}(A)$ or $\text{Nul}(A)$.)