The Four Subspaces

Recall: To any matrix A, we can associate:

- $\mathrm{Col}(A)$; basis = pivot columns of A
- $\mathrm{Nul}(A)$; basis = vectors in the PVF of $Ax=0$

There are two more subspaces: just replace A by A^T, then take Col & Nul.

Why? Next week...

Def: The row space of A is $\mathrm{Row}(A) = \mathrm{Col}(A^T)$.

This is the subspace spanned by the rows of A, regarded as vectors in $\mathbb{R}^n$.

This is a subspace of $\mathbb{R}^n$ $\quad n = \#$ columns

($n = \#$ entries in each row)

$\rightsquigarrow$ row picture

Eg: $$\mathrm{Row}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$$
$$= \mathrm{Col}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Fact: Row operations do not change the row space.

Why? If the rows are v_1, v_2, v_3 then $\mathrm{Row}(A) = \mathrm{Span}\{v_1, v_2, v_3\}$. Row ops:

- $R_1 \longleftrightarrow R_3$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_3, v_2, v_1\}$
- $R_2 \times\!= 3$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, 3v_2, v_3\}$
- $R_2 \mathrel{+}= 2R_1$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2 + 2v_1, v_3\}$

  because $v_2 + 2v_1 \in \mathrm{Span}\{v_1, v_2, v_3\}$

  and $v_2 = (v_2 + 2v_1) - 2v_1 \in \mathrm{Span}\{v_1, v_2 + 2v_1, v_3\}$ ✓

This is a col space (of A^T), so you know how to compute a basis (pivot columns of A^T). But you can also find a basis by doing elimination on A:

Thm: The nonzero rows of a REF of A form a basis for $\mathrm{Row}(A)$.

Eg:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix} \right\}$

Proof: by "forward-substitution":

(1) **Spans:** row ops don't change $\mathrm{Row}(A)$, and you can always delete the zero vector without changing the span

(2) **LI:**
$$0 = x_1\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + x_2\begin{pmatrix}0\\0\\-3\\-3\end{pmatrix} = \begin{pmatrix}x_1\\2x_1\\2x_1-3x_2\\x_1-3x_2\end{pmatrix}$$

● = pivot, so this entry in the sum is just
$$(1)\cdot x_1 = 0 \Rightarrow x_1 = 0$$

● = pivot, so this entry in the sum is just
$$(-3)x_2 = 0 \Rightarrow x_2 = 0$$

✓

Consequence: $\dim \mathrm{Row}(A) = \#$ pivot rows $= \#$ pivots $=$ rank.

(a nonzero row of an REF matrix has a pivot)

Def: The left null space of A is $\mathrm{Nul}(A^T)$.

This is the solution set of $A^Tx = 0$.

Notation: just $\mathrm{Nul}(A^T)$ (no new notation)

This is a subspace of $\mathbb{R}^m$ $\quad m = \#$ rows

($m = \#$ columns of A^T)

$\rightsquigarrow$ column picture

NB: $A^Tx = 0 \iff 0 = (A^Tx)^T = x^TA$

so $\mathrm{Nul}(A^T) = \{$ row vectors $y \in \mathbb{R}^m$: $yA = 0\}$

$\mathrm{Nul}(A^T)$ is a null space, so you know how to compute a basis (PVF of $A^Tx=0$). You can also find a basis by doing elimination on A:

Thm: If $EA = U$ for E an invertible $m \times m$ matrix and U a matrix in REF, and if U has $m-r$ zero rows, then the last $m-r$ rows of E form a basis for $\mathrm{Nul}(A^T)$.

r = rank = # pivots = # nonzero rows

Consequence: $\dim \mathrm{Nul}(A^T) = m - r$ = # rows − rank

Where did E come from? Elementary matrices! Doing row ops means left-multiplying by these:

$$A \rightsquigarrow E_1A \rightsquigarrow E_2E_1A \rightsquigarrow E_3E_2E_1A = U$$

So $EA = U$ for $E = E_3E_2E_1$, which is the matrix you get by doing the same row ops on I_m.

Procedure: To compute a basis of $\mathrm{Nul}(A^T)$:

(1) Form the augmented matrix $(A \mid I_m)$

(2) Eliminate to REF

(3) The rows on the right side of the line corresponding to zero rows on the left form a basis of $\mathrm{Nul}(A^T)$.

Eg: $A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix}$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 -= R_1]{R_2 -= 2R_1} \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 -= R_2} \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

zero row   basis

Basis for $\mathrm{Nul}(A^T)$: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Check: $\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ ✓

Proof of the Thm: If U is in REF and the last $m-r$ cols are zero then

$$\mathrm{Nul}(U^T) = \mathrm{Span}\{e_{m-r+1}, e_{m-r+2}, \ldots, e_m\}:$$

This is because $U^T e_i$ = the i^{th} row of U. We know from before that the nonzero rows of U are LI. And $U^T e_{m-r+i}$ = a zero row, so $e_{m-r+i} \in \mathrm{Nul}(U^T)$.

But $U = EA$, so $U^T = A^T E^T$, and

$$0 = U^T e_{m-r+i} = A^T E^T e_{m-r+i}$$
$$= A^T(m-r+i^{th} \text{ row of } E).$$

NB: The left null space is changed by row operations:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \qquad \mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$$\Big\updownarrow$$

$$U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{Nul}(U^T) = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Summary: Four Subspaces

A : an $m \times n$ matrix of rank r

Subspace	of	row/col	dim	basis
$\mathrm{Col}(A)$	$\mathbb{R}^m$	col	r ← # pivot cols	pivot cols of A
$\mathrm{Nul}(A)$	$\mathbb{R}^n$	row	$n-r$ ← # free vars	vectors in PVF
$\mathrm{Row}(A)$	$\mathbb{R}^n$	row	r ← # pivot rows	nonzero rows of REF
$\mathrm{Nul}(A^T)$	$\mathbb{R}^m$	col	$m-r$ ← # zero rows in REF	last $m-r$ rows of E

The row picture subspaces ($\mathrm{Nul}(A)$, $\mathrm{Row}(A)$) are unchanged by row operations.

The col picture subspaces ($\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$) are changed by row operations.

Consequences:

$$\boxed{\begin{array}{c}\text{Row Rank = Column Rank} \\ \dim \operatorname{Row}(A) = \operatorname{rank} = \dim \operatorname{Col}(A)\end{array}}$$

So A & A^T have the same # pivots — in completely different positions! (HW#4.19)

$$\boxed{\begin{array}{c}\text{Rank-Nullity} \\ \dim \operatorname{Col}(A) + \dim \operatorname{Nul}(A) = n = \#\text{cols} \\ \dim \operatorname{Row}(A) + \dim \operatorname{Nul}(A^T) = m = \#\text{rows}\end{array}}$$

[demos]

NB: You can compute bases for all four subspaces by doing elimination once.

$$A \rightsquigarrow [A \mid I_m] \rightsquigarrow [\operatorname{RREF}(A) \mid E]$$

- Get the pivots of $A \rightsquigarrow \operatorname{Col}(A)$
- Get $\operatorname{RREF}(A) \rightsquigarrow$ PVF of $Ax=0 \rightsquigarrow \operatorname{Nul}(A)$
- Get nonzero rows of $\operatorname{RREF}(A) \rightsquigarrow \operatorname{Row}(A)$
- Get rows of $E \rightsquigarrow \operatorname{Nul}(A^T)$

Full-Rank Matrices

Now we can see how the rank of a matrix can control its properties.

Def: An $m \times n$ matrix A of rank r has:

- full column rank if $r = n$
- full row rank if $r = m$

Thm: The Following Are Equivalent

(all are true for a given matrix A, or all false)

(1) A has full column rank

(2) A has a pivot in every column

(3) A has no free columns.

(4) $\mathrm{Nul}(A) = \{0\}$

(5) $Ax = 0$ has only the trivial solution.

☆ (6) $Ax = b$ has 0 or 1 soln for every $b \in \mathbb{R}^m$

(7) The columns of A are LI

(8) $\dim \mathrm{Col}(A) = n$

(9) $\dim \mathrm{Row}(A) = n$

Thm: TFAE:

(1) A has full row rank

(2) A has a pivot in every row

(3) A REF of A has no zero rows

(4) $\dim \mathrm{Col}(A) = m$

(5) $\mathrm{Col}(A) = \mathbb{R}^m$

(6) $Ax = b$ is consistent for every $b \in \mathbb{R}^m$

(7) The columns of A span $\mathbb{R}^m$

(8) $\dim \mathrm{Row}(A) = m$

(9) $\mathrm{Nul}(A^T) = \{0\}$

For a square matrix, these are the same as $m = r = n$, which means invertibility:

Thm: Let A be an $n \times n$ matrix. TFAE:

(1) A is invertible

(2) A has full column rank

(3) A has full row rank

(4) $\mathrm{RREF}(A) = I_n$

(5) There is a matrix B with $AB = I_n$

(6) There is a matrix B with $BA = I_n$

(7) A^T is invertible

Eg: If A is invertible then its columns

$$\left.\begin{array}{l} \bullet \text{ span } \mathbb{R}^n \text{ (full row rank)} \\ \bullet \text{ are LI (full col rank)} \end{array}\right\} \Rightarrow \text{basis for } \mathbb{R}^n$$

Conversely, any basis for $\mathbb{R}^n$ are the columns of an invertible matrix

- spans $\Rightarrow$ full row rank
- LI $\Rightarrow$ full col rank

$$\boxed{\text{Basis of } \mathbb{R}^n \equiv \text{cols of an invertible } n\times n \text{ matrix}}$$

So $\mathbb{R}^n$ has many bases! (not just $\{e_1, \dots, e_n\}$)

NB: for an $n\times n$ matrix,

full col rank $\Leftrightarrow$ invertible $\Leftrightarrow$ full row rank

In terms of colums, n vectors in $\mathbb{R}^n$

spans $\mathbb{R}^n \Leftrightarrow$ linearly independent

This is a special case of the basis theorem.

Basis Theorem: Let V be a subspace of dim d

(1) If d vectors span V then they're a basis

(2) If d vectors in V are LI then they're a basis.

So if you have the correct number of vectors, you only need to check one of spans / LI.

Eg: • Two noncollinear vectors in a plane form a basis.

• Two vectors that span a plane form a basis.