Orthogonal Complements

We are now aiming to find the "best" approximate solution of $Ax=b$ when no actual solution exists.

Eg: find the best-fit ellipse through these points from the 1st lecture...

Q: How close can Ax get to b?

$$\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^n\}$$

so this means: what is the closest vector $\hat{b}$ in $\mathrm{Col}(A)$ to b?

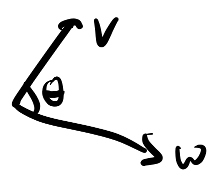
A: $b - \hat{b}$ is perpendicular to $\mathrm{Col}(A)$ [demo]

So we want to understand what vectors are perpendicular to a subspace.

Recall: v is orthogonal to w if $v \cdot w = 0$

Notation: $v \perp w$

NB: $v \cdot w = \|v\| \cdot \|w\| \cdot \cos(\theta)$



So this means $\theta = \pm 90°$

NB: $v \cdot w = v^T w$: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

NB: $0 \perp v$ for any v: $0 \cdot v = 0$

Eg: Find all vectors orthogonal to $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

We need to solve $v \cdot x = 0$

$$\iff v^T x = 0$$

This is just $\mathrm{Nul}(v^T)$:

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \rightsquigarrow x_1 + x_2 + x_3 = 0$$

$$\overset{\text{PF}}{\rightsquigarrow} \quad \begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= \quad x_2 \\ x_3 &= \qquad\quad x_3 \end{aligned}$$

$$\overset{\text{PVF}}{\rightsquigarrow} \quad x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

[demo] $\rightsquigarrow \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a plane

Check: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ ✓

Eg: Find all vectors orthogonal to $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ & $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

We need to solve $\begin{cases} v_1^T \cdot x = 0 \\ v_2^T \cdot x = 0 \end{cases}$

Equivalently, $\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix} \cdot x = \begin{pmatrix} v_1^T x \\ v_2^T x \end{pmatrix} = 0$

So we want $\text{Nul}\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{PF}} \begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned}$$

$$\xrightarrow{\text{PVF}} x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \text{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$ a line

Check: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ ✓

[demo]

NB: If $x \perp v_1$ and $x \perp v_2$ then

$$x \cdot (av_1 + bv_2) = a\, x \cdot v_1 + b\, x \cdot v_2 = a \cdot 0 + b \cdot 0 = 0$$

So x is orthogonal to every vector in $\text{Span}\{v_1, v_2\}$ [demo again]

More generally,

$$\boxed{\left\{ v \in \mathbb{R}^n : \begin{array}{l} v \text{ is orthogonal} \\ \text{to every vector} \\ \text{in } \mathrm{Span}\{v_1, \dots, v_n\} \end{array} \right\} = \mathrm{Nul}\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}}$$

Def: Two subspaces V, W of $\mathbb{R}^n$ are orthogonal if every vector in V is orthogonal to every vector in W:

$$v \in V \quad w \in W \implies v \cdot w = 0$$

The orthogonal complement of V is

$$V^\perp = \left\{ w \in \mathbb{R}^n : \begin{array}{l} w \text{ is orthogonal to} \\ \text{every vector in } V \end{array} \right\}$$

NB: Note the difference in notations:

- $V^\perp$ is the orthogonal complement of a subspc
- A^T is the transpose of a matrix.

Eg: $V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \implies V^\perp = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$

Eg: $V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\} \implies V^\perp = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

NB: If V & W are orthogonal and x is in both V and W then $x \cdot x = 0 \implies x = 0$

Translation of the red box above:

$$\boxed{\mathrm{Span}\{v_1,\dots,v_n\}^\perp = \mathrm{Nul}\begin{pmatrix} -v_1^T- \\ \vdots \\ -v_n^T- \end{pmatrix}}$$

Eg: $\{0\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{0\}$

NB: $V^\perp$ is also a subspace of $\mathbb{R}^n$.

Every subspace is a span, and the orthogonal complement of a span is a null space (which is a subspace).

NB: $V = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ and $W = \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$ are orthogonal subspaces, but W is not the orthogonal complement of V:

$$V^\perp = \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\} \neq \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}.$$

Facts: Let V be a subspace of $\mathbb{R}^n$.

(1) $\dim(V) + \dim(V^\perp) = n$

(2) $(V^\perp)^\perp = V$

[demos]

(2) says V and $V^\perp$ are orthogonal complements of each other.

Orthogonality of the Four Subspaces

Recall: If someone gives you a subspace, **Step 0** is to write it as a column space or a null space. So we want to understand $\mathrm{Col}(A)^\perp$ & $\mathrm{Nul}(A)^\perp$.

Let $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. Then

$$\mathrm{Col}(A)^\perp = \mathrm{Span}\{v_1, \dots, v_n\}^\perp = \mathrm{Nul}\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix} = \mathrm{Nul}(A^T)$$

$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$$

$$\xrightarrow{\text{Take } (\cdot)^\perp} \quad \mathrm{Col}(A) = (\mathrm{Col}(A)^\perp)^\perp = \mathrm{Nul}(A^T)^\perp$$

$$\xrightarrow[\text{by } A^T]{\text{replace } A} \quad \mathrm{Row}(A) = \mathrm{Col}(A^T) = \mathrm{Nul}(A)^\perp$$

and $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$

Orthogonality of the Four Subspaces:

$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$	$\mathrm{Nul}(A^T)^\perp = \mathrm{Col}(A)$
$\mathrm{Nul}(A)^\perp = \mathrm{Row}(A)$	$\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$

This says the two row picture subspaces $\mathrm{Row}(A)$, $\mathrm{Nul}(A)$ are orthogonal complements, & the two column picture subspaces $\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$ are orthogonal complements.

Eg: $V = \left\{ x \in \mathbb{R}^3 : \begin{matrix} x+2y=z \\ x+y+z=0 \end{matrix} \right\}$ What is $V^\perp$?

Step 0: $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow V^\perp = \mathrm{Row}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$V^\perp = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$: no work needed!

Eg: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$

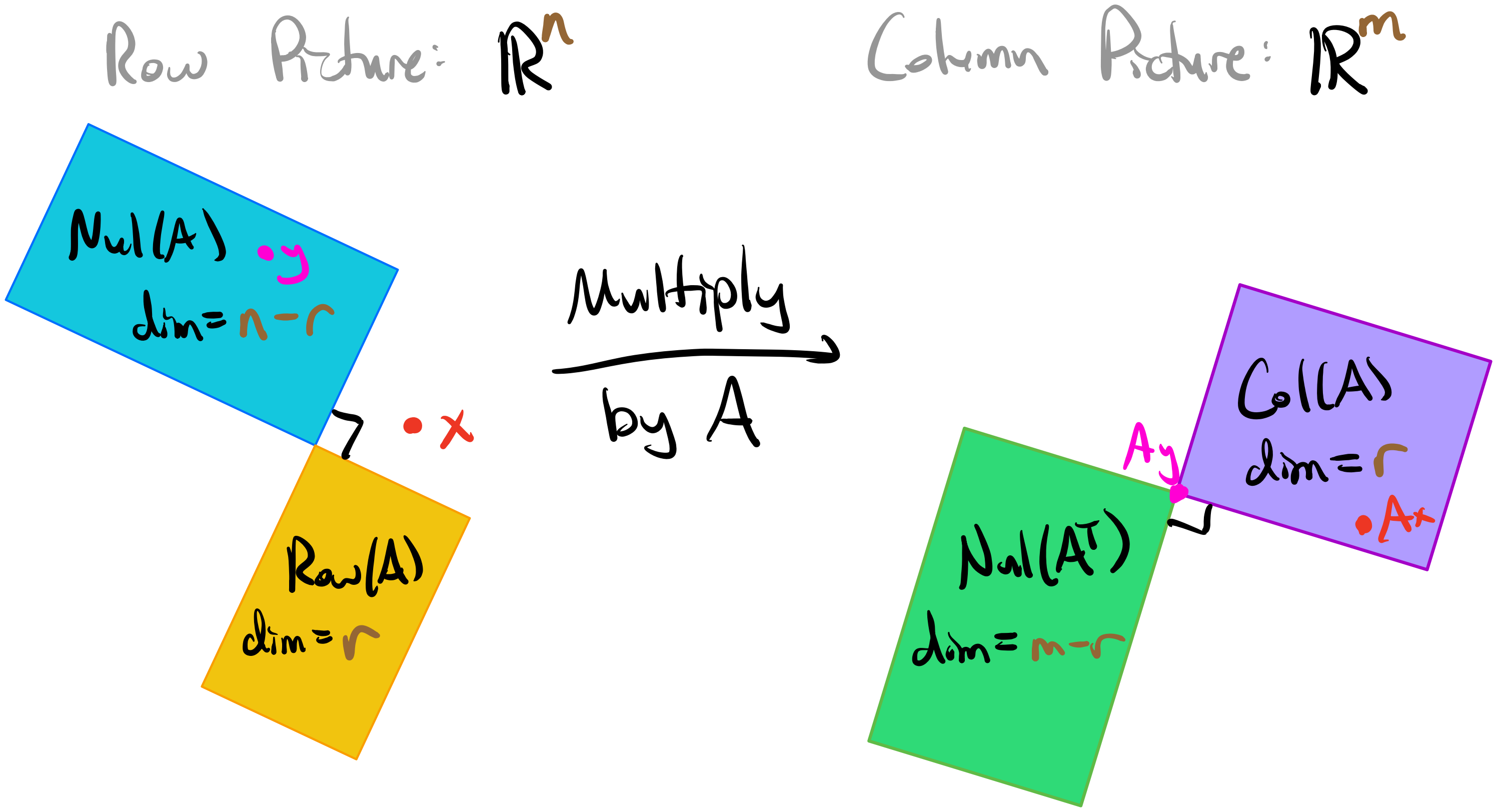
$\rightsquigarrow \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Nul}(A^T) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Here's the picture to have in your head:

The Big Picture

for an $m \times n$ matrix A of rank r



NB: The dimensions match up with $\dim V + \dim V^{\perp} = n$:

$$\dim \mathrm{Nul}(A) + \dim \mathrm{Row}(A) = n$$
$$\dim \mathrm{Nul}(A^T) + \dim \mathrm{Col}(A) = m \quad \checkmark$$

As an application, we can prove an Important Fact that we will use many times:

$$\boxed{\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)}$$

Proof: $\mathrm{Nul}(A^TA)$ contains $\mathrm{Nul}(A)$:

$$x \in \mathrm{Nul}(A) \Rightarrow Ax = 0 \Rightarrow A^TAx = 0 \Rightarrow x \in \mathrm{Nul}(A^TA)$$

$\mathrm{Nul}(A)$ contains $\mathrm{Nul}(A^TA)$:

$$x \in \mathrm{Nul}(A^TA) \Rightarrow A^TAx = 0 \Rightarrow Ax \in \mathrm{Nul}(A^T)$$

$$\Rightarrow Ax \in \mathrm{Col}(A) \text{ and } \mathrm{Nul}(A^T)$$

$$\Rightarrow (Ax)\cdot(Ax) = 0 \Rightarrow Ax = 0 \Rightarrow x \in \mathrm{Nul}(A)$$ ✓

NB: If A has columns $v_1, \ldots, v_n$ then

$$A^TA = \begin{pmatrix} — v_1^T — \\ \vdots \\ — v_n^T — \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & \cdots & v_1\cdot v_n \\ v_2\cdot v_1 & v_2\cdot v_2 & \cdots & v_2\cdot v_n \\ \vdots & & \ddots & \vdots \\ v_n\cdot v_1 & v_n\cdot v_2 & \cdots & v_n\cdot v_n \end{pmatrix}$$

This is the matrix of column dot products

Implicit Equations, Revisited

Recall: $\mathrm{Nul}(A) \overset{\text{PVF}}{\rightsquigarrow} \mathrm{Span}\{v_1, \ldots, v_{n-r}\}$

takes the implicit equation $Ax=0$

and generates the parametric form

$$x = a_1v_1 + \cdots + a_{n-r}v_{n-r}.$$

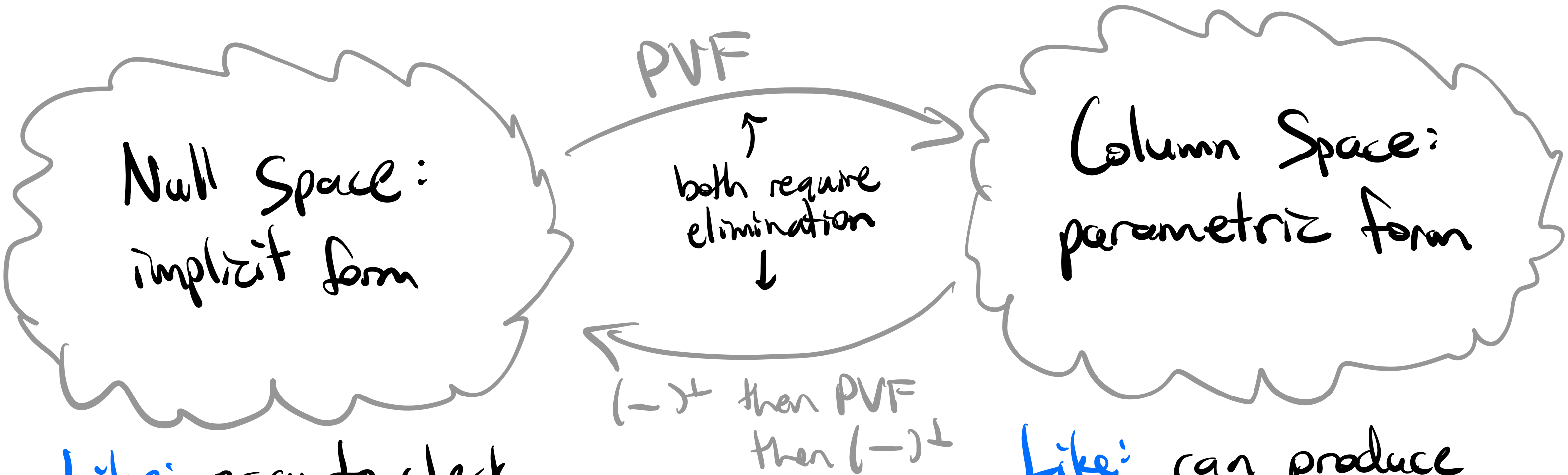
Orthogonal complements let us go the other way!

parametric $\rightarrow \mathrm{Col}(A) = \mathrm{Nul}(A^T)^\perp$

$$\mathrm{Nul}(A^T) \xrightarrow{\text{PVF}} \mathrm{Span}\{v_1, \dots, v_{m-r}\}$$

$$\Rightarrow \mathrm{Col}(A) = \mathrm{Span}\{v_1, \dots, v_{m-r}\}^\perp$$

$$= \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ \vdots \\ — v_{m-r}^T — \end{pmatrix} \leftarrow \text{implicit}$$



Like: easy to check if $x \in V$: $Ax = 0$

Like: can produce vectors in V: $x = a_1 v_1 + \dots + a_n v_n$

Eg: Find an implicit equation for the plane

$$V = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$$

$$V^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix} \underset{\text{p3}}{\xrightarrow{\text{PVF}}} \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$$

$$\Rightarrow V = \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}^\perp = \mathrm{Nul}\begin{pmatrix}-1 & 1 & 0\end{pmatrix}$$

$$= \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} : -x_1 + x_2 = 0\right\}$$