Determinants

What we've done:

- Solve $Ax=b$
  (Gauss-Jordan, LU, bases, ...)
- Approximately solve $Ax=b$
  (orthogonality, projections, QR, ...)

What's next:

- Solve $Ax=\lambda x$

This is the eigenvalue problem used in difference equations (rabbit population) & ODEs. It deals exclusively with square matrices.

The determinant of a square matrix is a number that satisfies many magical properties. I'll define it by telling you how to compute it using row operations.

→ Next time: other ways to compute it.

Def: The determinant of a square matrix A is a number $\det(A)$ or $|A|$ satisfying:

(1) If $A \xrightarrow{R_i += cR_j} B$ then $\det(A) = \det(B)$.

(2) If $A \xrightarrow{R_i \times = c} B$ then $\det(A) = \frac{1}{c}\det(B)$.

(3) If $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $\det(A) = -\det(B)$

(4) $\det(I_n) = 1$.

Consequence: if A has a zero row then $\det(A) = 0$

Eg: $\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \underset{(2)}{\overset{R_3 \times = -1}{=\!=\!=}} -\det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Longrightarrow \det\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Consequence: if A is (upper/lower) triangular then $\det(A)$ = product of diagonal entries

$$\boxed{\det\begin{pmatrix}\text{triangular} \\ \text{matrix}\end{pmatrix} = \text{product of the diagonal entries}}$$

eg. REF

Eg: $\det\begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[R_3 \times = \frac{1}{c}]{\overset{(2)}{R_1 \times = \frac{1}{a} \;\; R_2 \times = \frac{1}{b}}} abc \det\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow[\text{replacements}]{\text{row } (1)} abc \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \overset{(4)}{=} abc$

What if $b=0$ though? (or a? or c?)

$\det\begin{pmatrix} a & * & * \\ 0 & 0 & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\text{replacements}]{\text{row } (1)} \det\begin{pmatrix} a & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$

$= 0 = a \cdot 0 \cdot c$ ✓

A REF matrix is triangular, so you can compute $\det(A)$ by Gaussian elimination!

Eg: $\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow[(3)]{R_1 \leftrightarrow R_2} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$

$\xrightarrow[(1)]{R_3 -= R_1} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[(1)]{R_3 += R_2} -\det\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

$= -2$

NB: You get the same number for $\det(A)$ nomatter which row operations you do!

Eg: $\det\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \overset{R_1 \leftrightarrow R_3}{\underset{(3)}{=\!=}} -\det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$$\overset{R_2 -= R_1}{\underset{(1)}{=\!=}} -\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \overset{R_3 -= R_2}{\underset{(1)}{=\!=}} -\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= -2 \checkmark$$

Gaussian elimination is the fastest general algorithm for computing the determinant of a matrix (with known entries).

Procedure: To compute $\det(A)$, run Gaussian elimination: $A \xrightarrow[\text{operations}]{\text{row}} U$. Then

$$\boxed{\det(A) = (-1)^{\#\text{ row swaps}} \cdot \frac{1}{\prod(\text{row scaling})} \prod\begin{pmatrix}\text{diagonal} \\ \text{entries of } U\end{pmatrix}}$$

NB: You don't need to do row scaling operations to run Gaussian elimination, so this term usually does not appear.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- If $a \neq 0$:

$$\det(A) \xlongequal{R_2 -= \frac{c}{a}R_1} \det\begin{pmatrix} a & b \\ 0 & d-\frac{c}{a}b \end{pmatrix}$$

$$= a\left(d - \frac{c}{a}b\right) = ad - bc$$

- If $a = 0$:

$$\det(A) \xlongequal{R_1 \leftrightarrow R_2} -\det\begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

$$= -bc = ad - bc$$

$$\boxed{\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc}$$

NB: If U is a REF of A then

$\det(U) = \prod$ diagonal entries

$\det(U) \neq 0 \iff$ all diagonal entries are nonzero

$\iff U$ has n pivots

$\iff A$ has n pivots

$\iff A$ is invertible

Note $\det(A) = (\text{nonzero scalar}) \cdot \det(U)$, so

$$\boxed{A \text{ is invertible} \iff \det(A) \neq 0}$$

Magical Properties of the Determinant:

(1) Existence: There exists a number $\det(A)$ satisfying defining properties (1)–(4).

(2) Invertibility: A is invertible $\iff \det(A) \neq 0$

(3) Multiplicativity: $\det(AB) = \det(A)\det(B)$

and $\det(A) \neq 0 \implies \det(A^{-1}) = \dfrac{1}{\det(A)}$

(4) Transposes: $\det(A^T) = \det(A)$

We can think of $\det(-)$ as a function of n vectors (the cols of A):

$$\det\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \longleftrightarrow \det(v_1, \ldots, v_n)$$

(5) Multilinearity:

$$\det(v_1, \ldots, au + bw, \ldots, v_n)$$
$$= a\det(v_1, \ldots, u, \ldots, v_n) + b\det(v_1, \ldots, w, \ldots, v_n).$$

NB: (1) just says: You get the same number for $\det(A)$ nomatter which row ops you do!

Eg: $\det\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix} \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \det\begin{pmatrix}1&2&3\\0&-3&-6\\0&-6&-12\end{pmatrix}$

$\xrightarrow{R_3 -= 2R_2} \det\begin{pmatrix}1&2&3\\0&-3&-6\\0&0&0\end{pmatrix} = 0$ ● = pivots

so $\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$ is singular (not invertible)

NB: If the columns of A are linearly dependent then A does not have full column rank $\Rightarrow$ not invertible $\Rightarrow \det(A)=0$. Likewise for rows (take transposes).

$$\boxed{A \text{ has linearly dependent rows or columns} \Rightarrow \det(A)=0}$$

Eg: $\det\left[\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix}^{100}\right]$

$= \det\left[\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix}\cdot\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix}^{99}\right]$

$= \det\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix} \det\left[\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix}^{98}\right]$

$= \cdots = \left[\det\begin{pmatrix}0&1&3\\1&2&1\\1&1&0\end{pmatrix}\right]^{100} = (-2)^{100}$

More generally,

$$\boxed{\det(A^n) = \det(A)^n \quad \text{for all } n \geq 0 \text{ (and } n<0 \text{ if } \det(A) \neq 0)}$$

Eg: Say A has a $PA = LU$ decomposition.

$$\det(L) = \det\begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} = 1$$

You get P by doing row swaps on I_n, so

$$\det(P) = (-1)^{\#\text{row swaps}}$$

Hence

$$(-1)^{\#\text{row swaps}} \det(A) = \det(PA)$$

$$= \det(LU) = \det(L)\det(U)$$

$$= \det(U)$$

This recovers the formula on p.4 (we did no row scaling operations).

The transpose property says that $\det(A)$ satisfies (1)–(3) for column operations too: they're just row operations on A^T.

Eg:

$$
\begin{tikzcd}
\det\begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \arrow[r, equal, "C_1 -= 4C_3"] \arrow[d, equal, "\text{transpose}"] & \det\begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \arrow[d, equal, "\text{transpose}"] \\
\det\begin{pmatrix} 2 & 3 & 4 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \arrow[r, equal, "R_1 -= 4R_3"] & \det\begin{pmatrix} -14 & -9 & 0 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}
\end{tikzcd}
$$

So we can compute det using column ops:

$$
\det\begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \overset{C_1 -= 4C_3}{=\!=} \det\begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}
$$

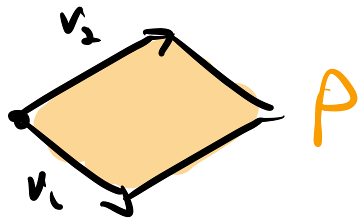
$$
\overset{C_1 += 9C_2}{=\!=} \det\begin{pmatrix} 49 & 7 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}
$$

$$
= 49
$$

Determinants and Volumes

Where do properties (1)–(4) come from?

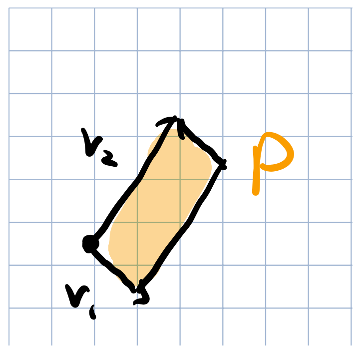
Two vectors $v_1, v_2 \in \mathbb{R}^2$ determine ("span") a parallelogram:



$$P = \{ x_1 v_1 + x_2 v_2 : x_1, x_2 \in [0,1] \}$$

Fact: $\operatorname{area}(P) = \left| \det \begin{pmatrix} - & v_1^T & - \\ - & v_2^T & - \end{pmatrix} \right|$

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

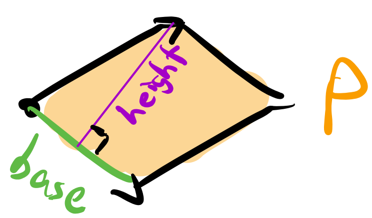


$$\operatorname{area}(P) = \left| \det \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \right|$$

$$= \left| (3)(1) - (2)(-1) \right| = 5$$

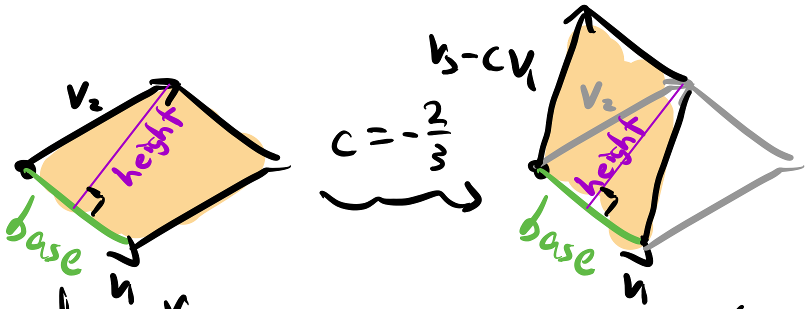
Why? Let's check that area(P) satisfies the four defining properties (1)–(4) of the determinant.

NB: area(P) = base × height:



(1) Row replacement

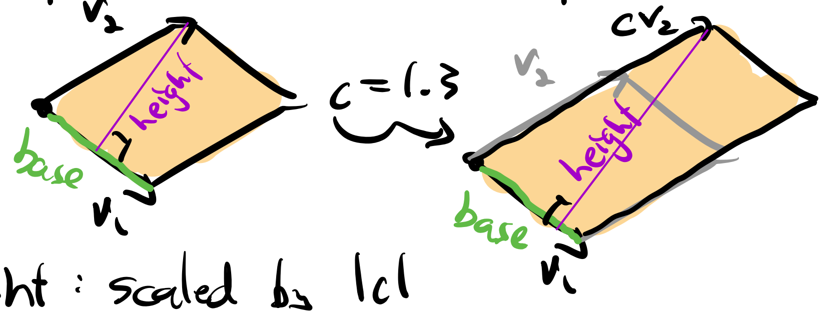
$v_2 \rightsquigarrow v_2 + cv_1$



area = base × ht : unchanged

(2) Row scaling

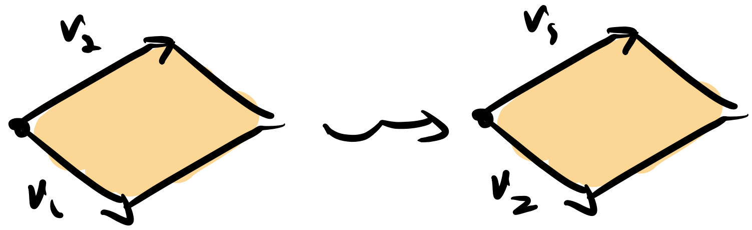
$v_2 \rightsquigarrow cv_2$



ht scaled by $|c| \Rightarrow$ base × ht : scaled by $|c|$

(3) Row Swap

$v_1 \longleftrightarrow v_2$



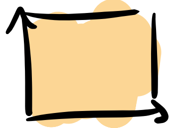
area unchanged $= |\det|$

↳ Q: minus sign?

HW 9

(4) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

area = 1.

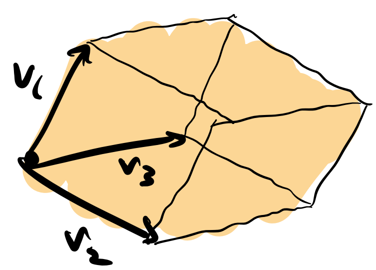


This generalizes as follows (same reasoning):

Def: The paralellepiped determined ("spanned") by n vectors $v_1, \dots, v_n \in \mathbb{R}^n$ is

$$P = \{x_1 v_1 + \cdots + x_n v_n : x_1, \dots, x_n \in [0,1]\}$$

P



Thm (Determinants & Volumes):

$$\text{volume}(P) = \left| \det \begin{pmatrix} — v_1^T — \\ \vdots \\ — v_n^T — \end{pmatrix} \right|$$

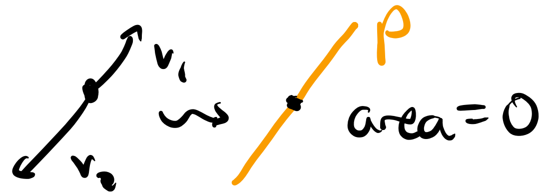
NB: When $n=1$, "volume" = "length":

$$\text{length}(a) = |a|$$

NB: When $n=2$, "volume" = "area".

Question: When is $\text{volume}(P)=0$?

When P is squashed flat:



ie when $v_1,\dots,v_n$ are linearly dependent ($\Rightarrow \det(\dots)=0$)

NB: In multivariable calc, you approximate shapes by tiny cubes, which turn into tiny parallelepipeds after applying a function. This is why determinants appear in the change of variables formula for integrals:

if $(y_1,\dots,y_n) = \varphi(x_1,\dots,x_n)$ then

$$dy_1 \cdots dy_n = \left|\det\left(\partial y_i / \partial x_j\right)\right| dx_1 \cdots dx_n$$