THE COFACTOR MATRIX

The purpose of this note is to justify the formula

$$AC^T = \det(A)I_n = C^T A,$$

where C is the cofactor matrix. You will not be tested on anything in this note.

For simplicity, we assume A is a 3×3 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

When we compute the product

$$AC^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix},$$

the $(1,1)$ entry is exactly

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13},$$

which computes $\det(A)$ by cofactor expansion along the first row of A. Likewise, the $(2,2)$ and $(3,3)$ entries are

$$a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \qquad a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33},$$

respectively, which compute $\det(A)$ by cofactor along the second and third rows. Hence the diagonal entries of AC^T are all equal to $\det(A)$.

It remains to show that the off-diagonal entries of AC^T are equal to zero. The $(1,2)$ entry is

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23},$$

which is the cofactor expansion along the second row of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

(the cofactors of this matrix along the second row equal the cofactors of A). Since this matrix has two identical rows, its determinant is zero. The other off-diagonal entries are zero for a similar reason, so we have shown that $AC^T = \det(A)I_n$.

For the identity $C^T A = \det(A)I_n$, we use the above identity as applied to A^T, noting that the cofactor matrix of A^T is just C^T:

$$A^T C = \det(A^T)I_n = \det(A)I_n.$$

Taking transposes of both sides gives $C^T A = \det(A)I_n$. (Alternatively, computing the entries of $C^T A$ corresponds to expanding cofactors along the *columns* of A.)