Math 218D Problem Session

Week 11

1. Matrices with complex eigenvalues

a) The eigenvalues of A are $1/2i = 1/2e^{\pi/2i}$ and $-1/2i = 1/2e^{-\pi/2i}$. These eigenvalues have $|\lambda| = 1/2$ and angle $\theta = \pm\pi/2$.

The eigenvalues of B are $(1+i) = \sqrt{2}e^{\pi/4i}$ and $(1-i) = \sqrt{2}e^{-\pi/4i}$.

b) For the matrix A, an eigenvector of $(1/2)i$ is $(1,-i)$, and an eigenvector for $-(1/2)i$ is $(1,i)$.

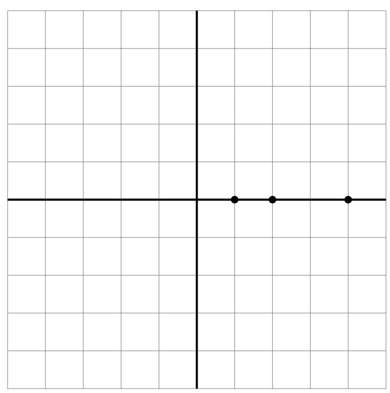
For the matrix B, an eigenvector $v_1 = (x_1, x_2)$ of B for the eigenvalue $\lambda_1 = (1+i)$ is a solution to $-ix_1 + x_2 = 0$, so we'll use $v_1 = (1,i)$ as the eigenvector.

An eigenvector for $\lambda_2 = \overline{\lambda_1}$ is $v_2 = \overline{v_1} = (1,-i)$.

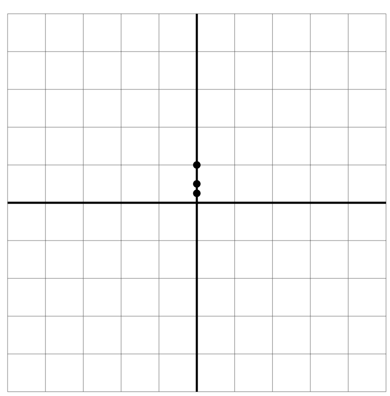
2. The dynamics of a diagonal matrix

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

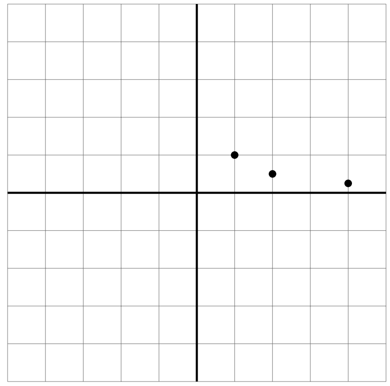
a) (1) $v = (1,0)$



(2) $v = (0,1)$



(3) $v = (1,1)$



b) We'll draw all the shapes on the same plot:

c) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches ∞, is $(1,0)$. We can see this from the picture, but can also compute this using limits. First,

$$\frac{A^n v}{||A^n v||} = \frac{2^n(1,0) + 2^{-n}(0,1)}{\sqrt{2^{2n} + 2^{-2n}}} = \frac{(1,0) + 2^{-2n}(0,1)}{\sqrt{1 + 2^{-4n}}},$$

where the second equality comes from dividing both the numerator and denominator by 2^n. Since $\lim_{n\to\infty}(1,0)+2^{-2n}(0,1) = (1,0)$ and $\lim_{n\to\infty}\sqrt{1+2^{-4n}} = 1$, we find that

$$\lim_{n\to\infty} \frac{A^n v}{||A^n v||} = \frac{(1,0)}{1} = (1,0).$$

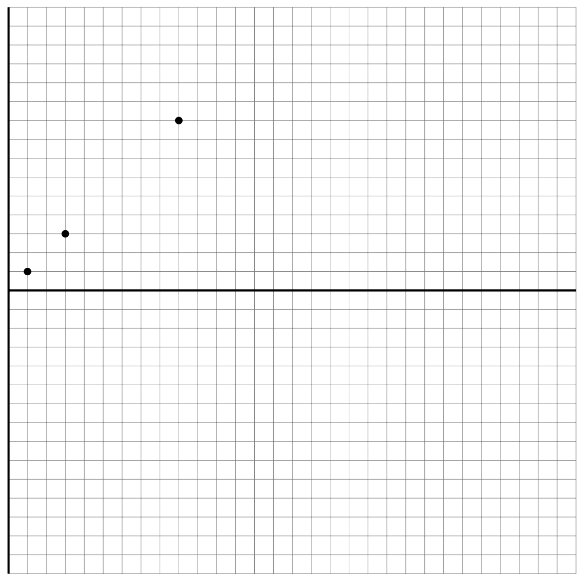
d) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches $-\infty$, is $(0,1)$.

3. The dynamics of a diagonalizable matrix

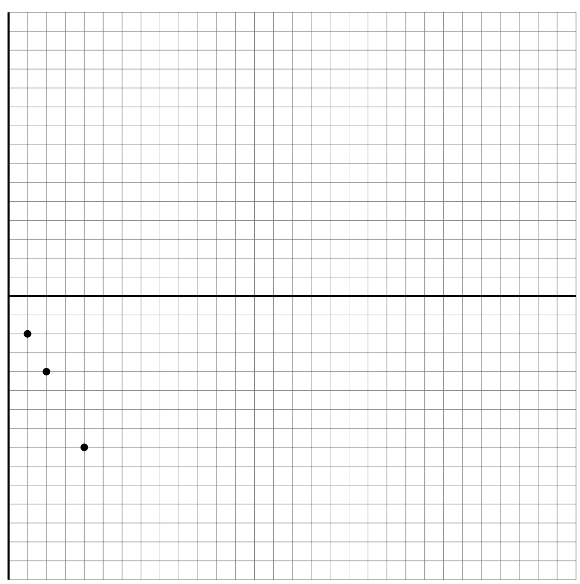
Consider the matrix A with $A(1,1) = 3(1,1)$ and $A(1,-2) = 2(1,-2)$. In other words, A is diagonalizable and you have been told the eigenvectors and eigenvalues.

a) For each of the following vectors, plot v, Av, A^2v:

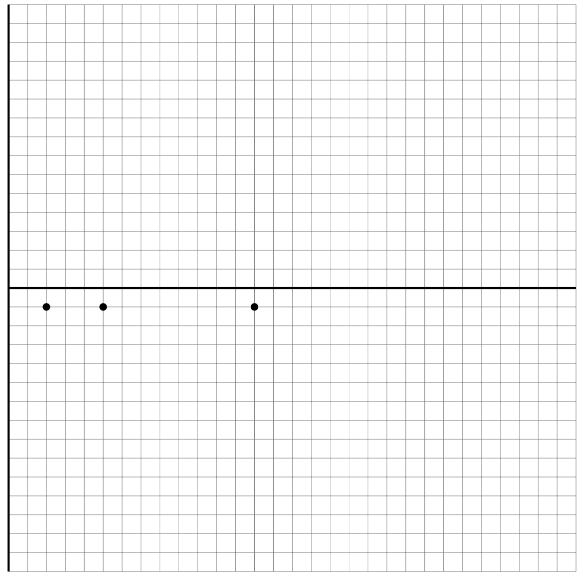
(1) $v = (1,1)$



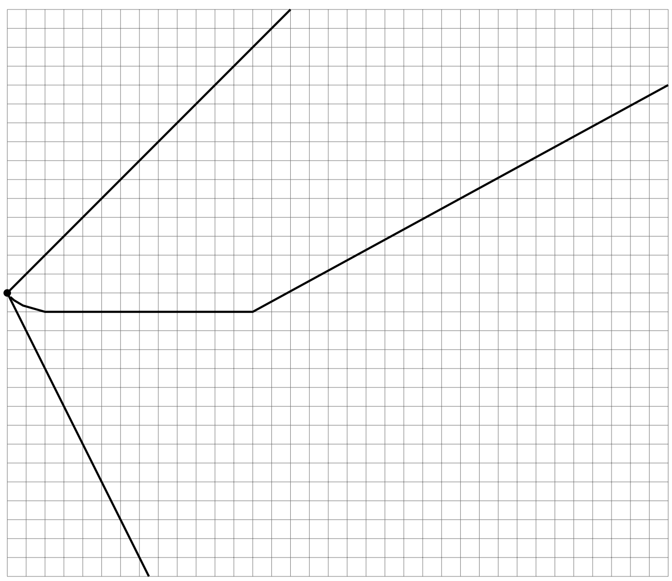
(2) $v = (1,-2)$

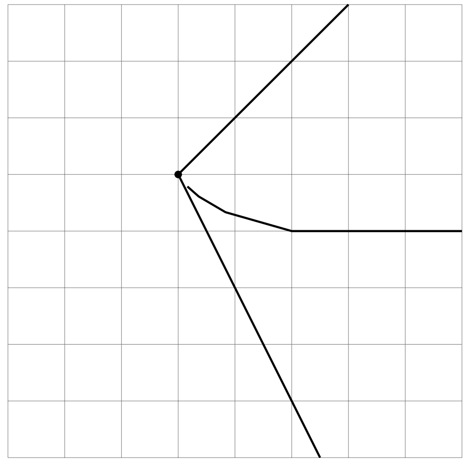


(3) $v = (2,-1)$



b) We'll draw all the shapes at once, once zoomed out and once zoomed it. The dot indicates the origin.





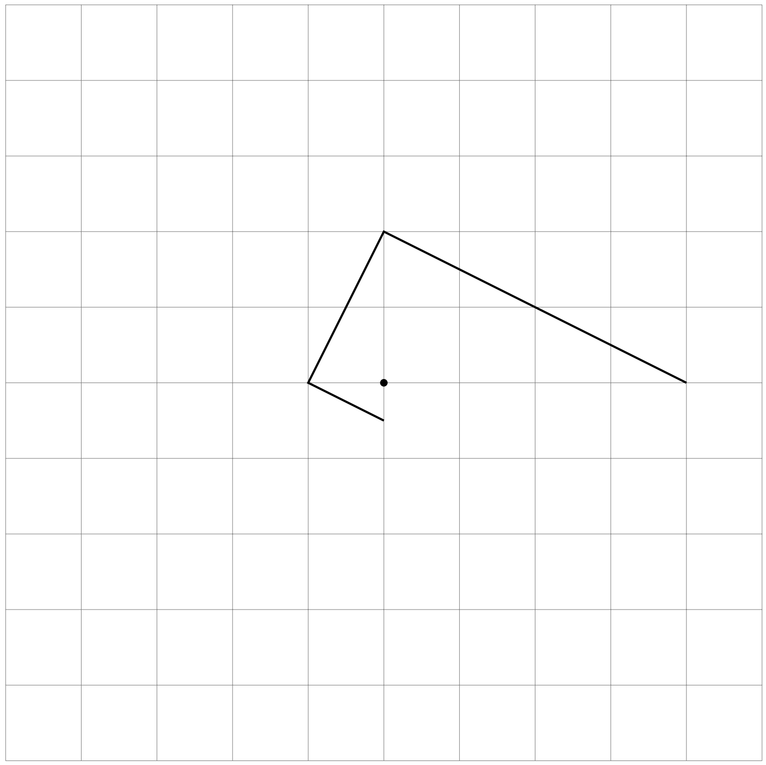
c) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches ∞, is $\frac{1}{\sqrt{2}}(1,1)$. This is not apparent from the pictures we drew, because we would need to zoom out much further to see this. But, as in Problem 1, the direction gets closer to the eigenspace of the larger eigenvalue.

d) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches $-\infty$, is $\frac{1}{\sqrt{5}}(1,-2)$. As in Problem 1, the direction gets closer to the eigenspace of the smaller eigenvalue.

4. Dynamics with complex eigenvalues

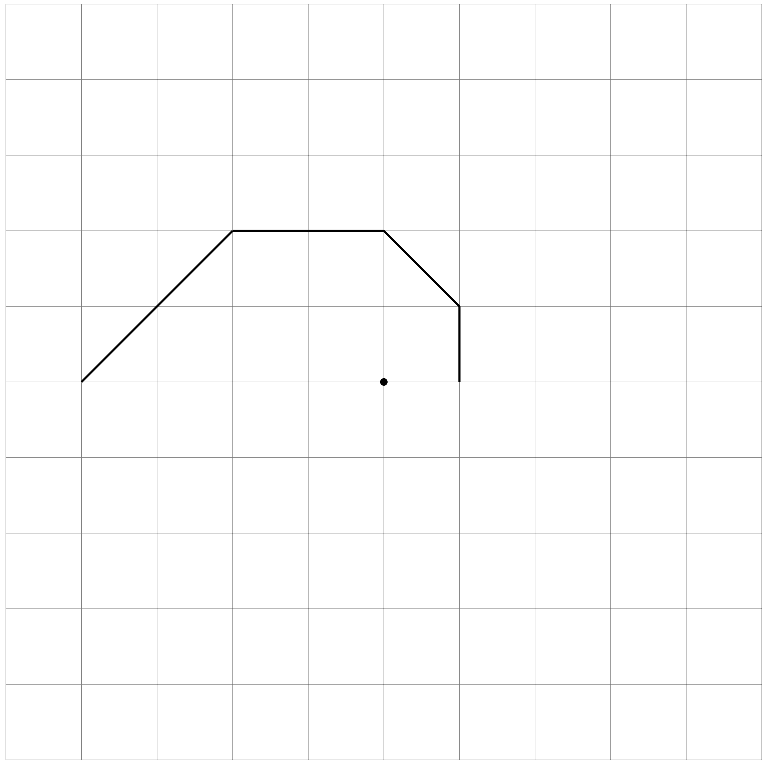
Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) The points $(4,0)$, $A(4,0)$, $A^2(4,0)$, $A^3(4,0)$, and $A^4(4,0)$:



The shape is a CCW spiral in towards the origin.

b) The points $(1,0)$, $B(1,0)$, $B^2(1,0)$, $B^3(1,0)$, and $B^4(1,0)$ (and also $B^5(1,0)$, as it helps see the pattern):



c) $(1,0) = \frac{1}{2}v_1 + \frac{1}{2}v_2$ (if you had a different choice of eigenvectors, these scalars might have been complex, but with our choice they are not.)

d) $B^n(1,0) = \frac{1}{2}\lambda_1^n v_1 + \frac{1}{2}\lambda_2^n v_2$.

e) We'll use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

The first component of $B^n(1,0)$ is

$$\frac{1}{2}\sqrt{2}^n(e^{n(\pi/4)i} + e^{-n(\pi/4)i}) = \frac{1}{2}\sqrt{2}^n(\cos(n\pi/4) + i\sin(n\pi/4) + \cos(-n\pi/4) + i\sin(-n\pi/4))$$
$$= \frac{1}{2}\sqrt{2}^n(2\cos(n\pi/4) + 0i) = \sqrt{2}^n\cos(n\pi/4).$$

(We used that that $\cos(-n\pi/4) = \cos(n\pi/4)$ while $\sin(n\pi/4) = -\sin(n\pi/4)$.)

The second component of $B^n(1,0)$ is

$$\frac{1}{2}\sqrt{2}^n(ie^{n(\pi/4)i} - ie^{-n(\pi/4)i}) = \frac{1}{2}\sqrt{2}^n(i\cos(n\pi/4) - \sin(n\pi/4) - i\cos(-n\pi/4) + \sin(-n\pi/4))$$
$$= \sqrt{2}^n\sin(n\pi/4).$$

Therefore the vector $B^n(1,0)$ has closed form

$$B^n(1,0) = (\sqrt{2}^n\cos(n\pi/4), \sqrt{2}^n\sin(n\pi/4)).$$

f) Based on the eigenvalues for A and the picture, we might guess

$$A^n(4,0) = (4(1/2)^n\cos(n\pi/2), 4(1/2)^n\sin(n\pi/2)).$$