Math 218D Problem Session

Week 4

1. Parallel lines

a) $\mathrm{Span}\{(-2,1,1)\}+(3,-2,1)$

b) One possible answer is $P_1=(3,-2,1)$, $P_2=(1,-1,2)$, $P_2-P_1=-(-2,1,1)$.

c) To get a parallel line, you need the same matrix A but a different b vector. You can find the correct b vector by multiplying A times $x=(1,1,1)$: $A(1,1,1)=(3,6)$. In other words, the solution set of

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + z &= 6 \end{aligned}$$

is parallel to L and passing through $(1,1,1)$.

2. The geometry of spans

a) No, it is not possible. You can confirm this by computing the RREF of $\left(\begin{array}{rr|r} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 5 & 1 & 0 \end{array}\right)$.

Alternately, you could observe that that the first two components of $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ add up to 0, while the first two components of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ do not.

b) It is all of $\mathbf{R}^3$, since $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is not contained in the plane $\mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$ (using 3a)).

c) By computing the REF of $\left(\begin{array}{rr|r} 1 & 1 & b_1 \\ -1 & -1 & b_2 \\ 5 & 1 & b_3 \end{array}\right)$, we confirm that the vectors $b=(b_1,b_2,b_3)$ which make

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

consistent are precisely those where $b_1+b_2=0$. This means that the plane parametrized by

$$x_1\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}+x_2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

has equation

$$x+y=0.$$

d) Yes, you can find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},$$

since $(4, -4, 0)$ solves the equation $x + y = 0$ found in 3c).

e) The vectors $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are not parallel, so they span a plane. The third vector $\begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$ is contained in that plane by 3d), so adding it to the list of vectors does not enlarge the span.

3. Subspaces?

a) Not a subspace, since it doesn't contain $(0, 0, 0)$.

b) A subspace, since it is the solution set of a homogeneous linear equation.

c) Not a subspace, since it doesn't contain $(0, 0, 0)$.

d) A subspace, since it is the left-null space of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$

e) Not a subspace, since it doesn't contain $(0, 0, 0)$.

f) A subspace, since

$$\{(x, y) \in \mathbf{R}^2 : x^2 + 2xy + y^2 = 0\} = \{(x, y) \in \mathbf{R}^2 : (x + y)^2 = 0\} = \{(x, y) \in \mathbf{R}^2 : x + y = 0\}.$$

4. The fundamental subspaces I

a) The Nul(A) and Nul(A^T) are points, while the Col(A) and Col(A^T) are all of $\mathbf{R}^2$.

b) $\dim(\text{Nul}(A)) + \dim(\text{Col}(A^T)) = 2$

5. The fundamental subspaces II

a) The spanning sets are $\text{Col}(A^T) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$, $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$, $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$, $\text{Nul}(A^T) = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$, although other answers are possible.

b) Draw the lines spanned by the vectors of **a)**.

c) $\dim(\text{Nul}(A)) + \dim(\text{Col}(A^T)) = 2$.

d) The lines Nul(A) and Col(A^T) are perpendicular. The lines Col(A) and Nul(A^T) are perpendicular.

6. The fundamental subspaces III

a) $\mathrm{Col}(A^T)$ is a subspace of $\mathbf{R}^3$

b) $\mathrm{Nul}(A)$ is a subspace of $\mathbf{R}^3$

c) $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^2$

d) $\mathrm{Nul}(A^T)$ is a subspace of $\mathbf{R}^2$

e) The column space is the line spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and the left-null space is the line spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

f) The subspace $\mathrm{Col}(A^T)$ is spanned by the vectors $(1,-1,2)$ and $(-2,2,-4)$, but these are scalar multiples of each other, so the row space is a line. The null space can be found via RREF: $\mathrm{rref}(A) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. The free variables are y and z, and the parametric form is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Therefore the null space is a plane in $\mathbf{R}^3$.

g) $\mathrm{Col}(A^T) = \mathrm{Span}\{(1,-1,2)\}$

h) $\mathrm{Nul}(A) = \mathrm{Span}\{(1,1,0),(-2,0,1)\}$

i) We consider the matrix $B = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, whose column space equals $\mathrm{Nul}(A)$. We find an equation for the column space of B by finding the REF of $\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 1 & 0 & b_2 \\ 0 & 1 & b_3 \end{array}\right)$, and finding the equation which makes the system consistent. The REF of this augmented matrix is

$$\left(\begin{array}{cc|r} 1 & -2 & b_1 \\ 0 & 2 & b_2 - b_1 \\ 0 & 0 & b_1 - b_2 + 2b_3 \end{array}\right).$$

The equation that (b_1, b_2, b_3) must satisfy to be in the column space of B (and hence the null space of A) is $b_1 - b_2 + 2b_3 = 0$. In other words, the equation for the plane $\mathrm{Nul}(A)$ is

$$x - y + 2z = 0.$$

j) The coefficients of the equation are $(1,-1,2)$. This is the same as the vector which spanned $\mathrm{Col}(A^T)$ (you may have gotten a scalar multiple of the vector spanning $\mathrm{Col}(A^T)$ instead.) The means that every vector in the plane is perpendicular to the vector $(1,-1,2)$, i.e. that the plane has *normal vector* $(1,-1,2)$. In other words, *the null space is orthogonal to the row space*. We will discuss the orthogonality of subspaces in more detail in Week 6.