Math 218D Problem Session

Week 6

1. Linear (in)dependence

a) Are the vectors $\binom{1}{0}$, $\binom{1}{-1}$ linearly independent? If not, write down a linear dependence relation.

b) Are the vectors $\binom{1}{1}$, $\binom{1}{-1}$, $\binom{3}{2}$ linearly independent? If not, write down a linear dependence relation.

c) What is the dimension of $\mathrm{Span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$? Why?

d) Consider 2 linearly independent vectors $u, v \in \mathbf{R}^n$. Show that the two vectors $u+v$, $u-v$ are linearly independent.

e) Consider 3 vectors $u, v, w \in \mathbf{R}^n$. Show that the three vectors $u+v$, $u+2v-w$, $v-w$ are linearly *dependent*.

f) Show that the vectors

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix}, \begin{pmatrix}2\\3\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

are linearly dependent, by writing down a linear dependence relation among them.

Hint: Write down the matrix A whose columns are these vectors, and find a non-zero vector in $\mathrm{Nul}(A)$. Why does this solve the question?

2. Bases from an LU decomposition

Suppose that you have an $A = LU$ decomposition, where

$$U = \begin{pmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but you don't know L or A.

a) Which of the subspaces $\mathrm{Row}(A)$, $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Nul}(A^T)$ can you find a basis for? If you *can* find a basis, do.

b) Which of the subspaces $\mathrm{Row}(A)$, $\mathrm{Col}(A)$, $\mathrm{Nul}(A)$, $\mathrm{Nul}(A^T)$ can you find the dimension of? If you *can* find a dimension, do.

3. Projection onto a line

For each of the following,

(1) project the vector b onto the line $V = \text{Span}\{v\}$;

(2) draw the three vectors $b, b_V, b_{V^\perp}$.

a) $b = (1, 1),\ v = (1, 0)$

b) $b = (0, 2),\ v = (1, 1)$

c) $b = (1, 2, 3),\ v = (1, 1, -1)$.

4. Planes and normal vectors

The subspace $V = \text{Span}\{(1,1,2),(1,3,1)\}$ of $\mathbf{R}^3$ is a plane.

a) Make the vectors $(1,1,2),(1,3,1)$ into the rows of a 2×3 matrix A - this means that $\text{Row}(A) = V$. Find a basis for $\text{Nul}(A)$. Since

$$V^{\perp} = \text{Row}(A)^{\perp} = \text{Nul}(A),$$

you have found a basis $v = (a,b,c)$ for the line $V^{\perp}$.

In other words, you have found a basis for $V^{\perp}$ by solving the two orthogonality equations

$$(a,b,c)\cdot(1,1,2) = a + b + 2c = 0,$$
$$(a,b,c)\cdot(1,3,1) = a + 3b + c = 0.$$

b) Confirm that V is the plane $\{(x,y,z) \in \mathbf{R}^3 : ax + by + cz = 0\}$, by showing that both $(1,1,2)$ and $(1,3,1)$ solve this equation. *The coefficients of a plane's equation make a normal vector for the plane.*

c) Find the orthogonal decomposition $b = b_V + b_{V^{\perp}}$ of the vector $b = (1,1,1)$ with respect to the plane V and the orthogonal line $V^{\perp}$.

Hint: It is easier to compute $b_{V^{\perp}}$, as it is the projection of b onto the line $V^{\perp}$ spanned by the vector $v = (a,b,c)$.

5. Projection onto a plane

Consider the plane

$$V = \text{Span}\{(1,1,1,1),(1,2,3,4)\}$$

in $\mathbf{R}^4$. We will find the orthogonal projection of $b = (1,-1,-3,-5)$ onto V. This is a vector $b_V \in \mathbf{R}^4$ so that $b_V \in V$ and $b_{V^\perp} = b - b_V \in V^\perp$.

Since b_V is in V, it must equal

$$b_V = \widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)$$

for some scalars $\widehat{x}_1$ and $\widehat{x}_2$. **We will compute the orthogonal projection by solving for these scalars.**

The vector $b_{V^\perp}$ is orthogonal to every vector in V, in particular it is orthogonal to both $(1,1,1,1)$ and $(1,2,3,4)$. We get two equations:

$$(1,1,1,1)\cdot b_{V^\perp} = 0,$$

$$(1,2,3,4)\cdot b_{V^\perp} = 0.$$

Expanding $b_{V^\perp} = b - b_V = (1,-1,-3,-5) - (\widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4))$, we can rewrite these two equations as

$$(1,1,1,1)\cdot(\widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)) = (1,1,1,1)\cdot(1,-1,-3,-5),$$

$$(1,2,3,4)\cdot(\widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)) = (1,2,3,4)\cdot(1,-1,-3,-5).$$

a) By computing the dot-products, convert this into two linear equations in the two unknowns $\widehat{x}_1$ and $\widehat{x}_2$.

b) Solve for $\widehat{x}_1$ and $\widehat{x}_2$, and compute the orthogonal projection

$$b_V = \widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4).$$

c) Confirm that the vector $b_{V^\perp} = b - b_V$ is orthogonal to V by checking that

$$b_{V^\perp}\cdot(1,1,1,1) = 0 \text{ and } b_{V^\perp}\cdot(1,2,3,4) = 0.$$

d) Write down a matrix A whose column are the two vectors which span V, and compute A^TA, the "matrix of dot products". Compute the vector A^Tb. Explain where the matrix equation $A^TA\widehat{x} = A^Tb$ (the **normal equation**) appears in **a)-b)**, and also where the product $b_V = A\widehat{x}$ appears.

e) Compute the projection matrix $P = A(A^TA)^{-1}A^T$ for the subspace V – this is the matrix which, when multiplied with b, produces the projection b_V:

$$Pb = b_v.$$

We'll discuss projection matrices in next week's lectures.

f) Compute the vectors $(I_4 - P)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $(I_4 - P)\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$. Explain why these two vectors give a basis for the plane $V^\perp$.

g) Use your answer to **f)** to describe the plane V via *two* implicit equations:

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : c_1x_1+c_2x_2+c_3x_3+c_4x_4 = 0 \text{ and } c_1'x_1+c_2'x_2+c_3'x_3+c_4'x_4 = 0\}.$$

In other words, what coefficient vectors (c_1, c_2, c_3, c_4) and (c_1', c_2', c_3', c_4') can we use to describe V, and why? Confirm that every vector in V satisfies these equations by checking that both $(1, 1, 1, 1)$ and $(1, 2, 3, 4)$ do.