Math 218D Problem Session

Week 6

1. Linear (in)dependence

a) Since neither vector is a scalar multiple of the other, the two vectors are linearly independent.

b) Any 3 vectors in $\mathbf{R}^2$ must be linearly dependent. To find a dependence, we will compute the null space of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}$. Since the RREF of A is $\begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & 1/2 \end{pmatrix}$, we find that $\begin{pmatrix} -5/2 \\ -1/2 \\ 1 \end{pmatrix}$ a vector in the null space. In other words, $-\frac{5}{2}\binom{1}{1} - \frac{1}{2}\binom{1}{-1} + \binom{3}{2} = 0$ is a linear dependence relation among these three vectors.

c) The dimension is the same as the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, since the rank of a matrix equals the dimension of its column space. The rank of this matix is 3 (you could compute its REF, or notice that the transpose A^T is already in REF).

d) Consider any two scalars a, b such that

$$a(u+v) + b(u-v) = 0.$$

We need to show that both of these scalars are in fact equal to 0 - this would show that no linear dependence relations between $u+v$ and $u-v$ are possible.

The first equation implies that $(a+b)u + (b-a)v = a(u+v) + b(u-v) = 0$. Since u and v are linearly independent, this implies that $a+b = 0$ and $b-a = 0$. You can solve these two equations to find $a = 0, b = 0$.

e) The vectors $u+v$, $u+2v-w$, $v-w$ are linearly dependent, since $(u+v)+(v-w) = u+2v-w$.

f) The matrix $A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & -4 & 1 & 1 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The columns of the RREF are dependent: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 0 \end{pmatrix} = 0$. The same dependence relation works for the original vectors (since RREF doesn't change the null space):

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}-\begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix}-\begin{pmatrix}2\\3\\0\\1\end{pmatrix}-4\begin{pmatrix}0\\0\\1\\1\end{pmatrix}=0.$$

2. Bases from an LU decomposition

Suppose that you have an $A = LU$ decomposition, where

$$U = \begin{pmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but you don't know L or A.

a) We can find a basis for Row(A) and Nul(A) - since row operations change Col(A) and Nul(A^T), we can't hope to find them using U. A basis for Row(A) comes from the non-zero rows of U:

$$(1, -1, 2, 3, 5),\ (0, 0, 1, 2, 2),\ (0, 0, 0, 01).$$

To find the null space basis, we finish putting A into RREF – its RREF is $\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The parametric equations for $Ax = 0$ are then

$$\begin{aligned} x_1 &= x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

and the parametric vector form is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. A basis for the null space Nul(A) is given by the two vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

b) We can find all of the dimensions: $\dim \text{Row}(A) = 3$ and $\dim \text{Nul}(A) = 2$ from part a). Since row rank equals column rank equals the number of pivots, $\dim \text{Col}(A) = 3$. Since $\dim \text{Col}(A) + \dim \text{Nul}(A^T) = \#$ of rows $= 4$, we find that $\dim \text{Nul}(A^T) = 1$.

3. Projection onto a line

a) $b = (1,1),\ v = (1,0)$.

$b_V = \frac{b\cdot v}{v\cdot v} v = 1(1,0) = (1,0)$. Then $b_{V^\perp} = b - b_V = (0,1)$.

b) $b = (0,2),\ v = (1,1)$.

$b_V = \frac{b\cdot v}{v\cdot v} v = \frac{2}{2} v = v = (1,1)$. Then $b_{V^\perp} = b - b_V = (-1,1)$.

c) $b = (1,2,3),\ v = (1,1,-1)$.

$b_V = \frac{b\cdot v}{v\cdot v} v = (0,0,0)$. Then $b_{V^\perp} = b - b_V = b$.

4. Planes and normal vectors

The subspace $V = \text{Span}\{(1,1,2),(1,3,1)\}$ of $\mathbf{R}^3$ is a plane.

a) The matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \end{pmatrix}$. The null space of A is spanned by $(-5/2, 1/2, 1)$. This is a basis for $V^{\perp}$.

b) The equation $-\frac{5}{2}x + \frac{1/2}{y} + z = 0$ is true for both $(x,y,z) = (1,1,2)$ and $(x,y,z) = (1,3,1)$.

c) $b = (1,1,1)$. We find $b_{V^\perp}$ first. Note that $V^{\perp}$ is spanned by $(-5,1,2) = 2(-5/2,1/2,1)$ - this will make the arithmetic a little easier. Then $b_{V^\perp} = \frac{(1,1,1)\cdot(-5,1,2)}{(-5,1,2)\cdot(-5,1,2)}(-5,1,2) = \frac{-2}{30}(-5,1,2) = -\frac{1}{15}(-5,1,2)$.

Then $b_V = b - b_{V^\perp} = (1,1,1) - (-\frac{1}{15}(-5,1,2)) = (10/15, 16/15, 17/15) = (2/3, 16/15, 17/15)$.

5. Projection onto a plane

a) Take the two equations

$$(1,1,1,1)\cdot(\widehat{x}_1(1,1,1,1)+\widehat{x}_2(1,2,3,4)) = (1,1,1,1)\cdot(1,-1,-3,-5),$$

$$(1,2,3,4)\cdot(\widehat{x}_1(1,1,1,1)+\widehat{x}_2(1,2,3,4)) = (1,2,3,4)\cdot(1,-1,-3,-5)$$

and compute all dot products (using the distributivity of dot products and addition). We get two equations

$$4\widehat{x}_1 + 10\widehat{x}_2 = -8,$$

$$10\widehat{x}_1 + 30\widehat{x}_2 = -30.$$

b) We solve these two equations by computing the RREF of $\left(\begin{array}{cc|c} 4 & 10 & -8 \\ 10 & 30 & -30 \end{array}\right)$, which is $\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right)$. Therefore $\widehat{x}_1 = 3, \widehat{x}_3 = -2$, and

$$b_V = 3(1,1,1,1) - 2(1,2,3,4) = (1,-1,-3,-5).$$

(The fact that $b = b_V$ is a bit of a coincidence.)

c) $b_{V^\perp} = b - b_V = 0$, so $b_{V^\perp}$ is orthogonal to V.

d) $A^T A = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$ and $A^T b = (-8,-30)$. The equation $A^T A\widehat{x} = A^T b$ is the same as the system of equations

$$4\widehat{x}_1 + 10\widehat{x}_2 = -8,$$

$$10\widehat{x}_1 + 30\widehat{x}_2 = -30$$

from **a)-b)**. The equation $b_V = A\widehat{x}$ is the same as the equation $b_V = \widehat{x}_1(1,1,1,1)+\widehat{x}_2(1,2,3,4)$.

e) The projection matrix is $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T$.

We compute the inverse:

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} = \frac{1}{120-100}\begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix}.$$

Then

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T$$
$$= \begin{pmatrix} 1 & -3/10 \\ 1/2 & -1/10 \\ 0 & 1/10 \\ -1/2 & 3/10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$
$$= \begin{pmatrix} 7/10 & 2/5 & 1/10 & -1/5 \\ 2/5 & 3/10 & 1/5 & 1/10 \\ 1/10 & 1/5 & 3/10 & 2/5 \\ -1/5 & 1/10 & 2/5 & 7/10 \end{pmatrix}.$$

This is tedious to compute by hand - I used I computer to multiply these.

f) The matrix $I_4 - P$ equals $\begin{pmatrix} 3/10 & -2/5 & -1/10 & 1/5 \\ -2/5 & 7/10 & -1/5 & -1/10 \\ -1/10 & -1/5 & 7/10 & -2/5 \\ 1/5 & -1/10 & -2/5 & 3/10 \end{pmatrix}$. The first two columns are the same as $(I_4 - P)(1,0,0,0)$ and $(I_4 - P)(0,1,0,0)$. The vectors $(3/10, -2/5, -1/10, 1/5)$ and $(-2/5, 7/10, -1/5, -1/10)$ are then a basis for $V^\perp$. Why? Recall that $P_{V^\perp} = I - P_V$, so both of these vectors are the result of projection onto $V^\perp$, and so are contained in $V^\perp$. Since $V^\perp$ is 2-dimensional (since V was 2-dimensional and $\dim(V) + \dim(V^\perp) = \dim(\mathbf{R}^4) = 4$), to check that these two vectors are a basis, we only need to check that they are not scalar multiples of each other, which is true.

g) We scale the vectors we found in the previous part by 10, to make them simpler, so that $\{(3, -4, -1, 2), (-4, 7, -2, -1)\}$ are a basis for $V^\perp$. We use these vectors to give equations for V:

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : 3x_1 - 4x_2 - x_3 + 2x_4 = 0 \text{ and } -4x_1 + 7x_2 - 2x_3 - x_4 = 0\}.$$

Both equations are satisfied by the vectors $(1,1,1,1)$ and $(1,2,3,4)$ — this confirms that we have found correct equations.