MATH 218D-1
PRACTICE FINAL EXAMINATION

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. You may use an 8.5×11" **note sheet** as well. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1. [15 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 6 & 1 & 4 \\ 1 & 11 & 1 \\ 4 & 1 & 6 \end{pmatrix}.$$

The eigenvalues of S are $2, 9$, and 12.

a) Orthogonally diagonalize S: find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

$$Q = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

b) Compute the quadratic form associated to S:

$$q(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 6 & 1 & 4 \\ 1 & 11 & 1 \\ 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
$$= 6x_1^2 + 11x_2^2 + 6x_3^2 + 2x_1x_2 + 8x_1x_3 + 2x_2x_3.$$

c) Find the maximum and minimum values of $q(x)$ subject to $\|x\| = 1$. At which vectors are these extrema achieved?

Minimum value: $\boxed{2}$ achieved at $\pm\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Maximum value: $\boxed{12}$ achieved at $\pm\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Problem 2.

[15 points]

Consider the following symmetric matrices:

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & -1 & 1 \\ 2 & 1 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & -6 & -3 \\ -6 & 13 & 9 \\ -3 & 9 & 16 \end{pmatrix} \qquad \begin{pmatrix} 4 & -5 & 2 \\ -5 & 0 & 11 \\ 2 & 11 & 14 \end{pmatrix}.$$

a) Which *one* of the above matrices is positive-definite? Circle your answer.

b) Compute the LDL^T and Cholesky decompositions of the matrix you selected in **a)**: $A = LDL^T = L_1 L_1^T$ for

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ -2\sqrt{3} & 1 & 0 \\ -\sqrt{3} & 3 & 2 \end{pmatrix}$$

Problem 3. [40 points]

A certain 3×3 matrix A has the singular value decomposition

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

where:

$$\sigma_1 = 2 \qquad u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} \qquad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$\sigma_2 = \frac{1}{2} \qquad u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}.$$

This problem is *much easier* if you do not compute A.

a) What is the rank of A? What is $\dim \mathrm{Nul}(A)$?

$$\mathrm{rank}(A) = \boxed{2} \qquad \dim \mathrm{Nul}(A) = \boxed{1}$$

b) Find orthonormal bases for all four fundamental subspaces of A.

$$\mathrm{Col}(A): \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} \right\} \qquad \mathrm{Nul}(A): \left\{ \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix} \right\}$$

$$\mathrm{Row}(A): \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix} \right\} \qquad \mathrm{Nul}(A^T): \left\{ \frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix} \right\}$$

c) Write the SVD of A in matrix form: $A = U\Sigma V^T$ for

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{pmatrix}$$

d) Write the SVD of A^T in outer product form.

$$A^T = \sigma_1 v_1 u_1^T + \sigma_2 v_2 u_2^T$$

e) What are the maximum and minimum values of $\|Ax\|$ subject to $\|x\| = 1$? At which vectors are these extrema achieved?

Minimum value: $\boxed{0}$ achieved at $\pm\frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$

Maximum value: $\boxed{2}$ achieved at $\pm\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$

f) What is the pseudo-inverse A^+ of A? (You can write it as a sum of products of matrices.)

$$A^+ = \frac{1}{\sigma_1} v_1 u_1^T + \frac{1}{\sigma_2} v_2 u_2^T$$

g) Compute $A^+(4u_1 - 3u_2)$. (Don't expand out your answer in **f)**.)

$$A^+(4u_1 - 3u_2) = 2v_1 - 6v_2$$

h) Find the orthogonal decomposition of $b = (3, 6, -3)$ with respect to $V = \text{Col}(A)$: $b = b_V + b_{V^\perp}$ for

$$b_V = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \qquad b_{V^\perp} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$$

Problem 4.

[10 points]

Consider the matrix

$$A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}.$$

a) Diagonalize A: find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

$$C = \begin{pmatrix} -1 & -3 \\ 1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

b) For which vectors x does $\|A^k x\|$ *not* approach ∞ as $k \to \infty$?

$$x \in \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Problem 5. [15 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 3 \\ 5 & 4 \end{pmatrix}.$$

a) Compute the singular value decomposition of A in matrix form: $A = U\Sigma V^T$ for

$$U = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \qquad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

b) (Unrelated to **a)**.) Is A diagonalizable over the real numbers? Why or why not?

Yes: it has two real eigenvalues $2 \pm \sqrt{19}$.

Problem 6. [20 points]

Short answer questions: you need not explain your answers. In each case, assume that the entries of all matrices are real numbers.

a) Which of the following are subspaces of $\mathbf{R}^3$? *Circle all that apply.*

$$\mathrm{Nul}\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix} \qquad \mathrm{Col}\begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 0 & 3 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

$$\text{The solution set of } \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 15 \\ 0 & 0 & 4 \end{pmatrix} x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{The 1-eigenspace of } \begin{pmatrix} 2 & 2 & 0 \\ 1 & 5 & -2 \\ 0 & 2 & -1 \end{pmatrix}.$$

b) Let A be a 3×4 matrix. Which of the following are possible? *Circle all that apply.*

(1) The equation $Ax = 0$ has only the trivial solution.

(2) $\mathrm{rank}(A) = \dim(\mathrm{Nul}(A))$.

(3) The equation $Ax = b$ is consistent for each $b \in \mathbf{R}^3$.

c) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = -2$. Find $\det(3A)$ for

$$A = \begin{pmatrix} -4a + d & -4b + e & -4c + f \\ a & b & c \\ g & h & i \end{pmatrix}.$$

$$\det(3A) = \boxed{54}$$

d) Let A be a 2×2 matrix with eigenvalue 5. Suppose that A is *not* diagonalizable. What is the characteristic polynomial of A?

$$p(\lambda) = (\lambda - 5)^2$$

e) Let A be a 3×3 matrix whose 3-eigenspace is Span $\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ and whose 1-eigenspace is Span $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$. Find $\det(A)$.

$$\det(A) = \boxed{9}$$

Problem 7. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 4 & 0 \\ 0 & 2 & -1 \end{pmatrix}.$$

a) Find the QR decomposition of A: $A = QR$ for

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{2} & 3\sqrt{2} & 2\sqrt{2} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

b) Find the LU decomposition of A: $A = LU$ for

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & -4 \\ 0 & 0 & 3 \end{pmatrix}$$

c) Which of the above decompositions would you use to solve $Ax = b$ for multiple values of b, and why?

Probably the LU decomposition, since it's easier to compute, but since the matrix is invertible, either decomposition would work.

Problem 8. [10 points]

Consider the following centered data points (five samples of three measurements each):

$$\begin{pmatrix} 0.843 \\ -2.725 \\ 1.862 \end{pmatrix} \quad \begin{pmatrix} -0.942 \\ -2.794 \\ -1.653 \end{pmatrix} \quad \begin{pmatrix} -0.117 \\ 7.972 \\ -0.171 \end{pmatrix} \quad \begin{pmatrix} 2.732 \\ 0.409 \\ 5.238 \end{pmatrix} \quad \begin{pmatrix} -2.515 \\ -2.862 \\ -5.275 \end{pmatrix}.$$

The covariance matrix is

$$S = \begin{pmatrix} 3.851 & 1.929 & 7.682 \\ 1.929 & 21.788 & 3.857 \\ 7.682 & 3.857 & 15.373 \end{pmatrix}.$$

The eigenvalues of S are $\lambda_1 = 25, \lambda_2 = 16$, and $\lambda_3 = 0.01$, with associated unit eigenvectors

$$u_1 = \begin{pmatrix} 0.267 \\ 0.802 \\ 0.535 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0.359 \\ -0.598 \\ 0.717 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 0.894 \\ 0 \\ -0.447 \end{pmatrix}.$$

a) What is the total variance of these data points?

$$s^2 = 25 + 16 + 0.01 = 41.01$$

b) The first and third measurements are (circle one):

correlated   anti-correlated   not correlated

c) These data fit well to the (circle one)

point   line   plane

spanned by

$$\{u_1, u_2\}.$$

because:

The total variance in the direction orthogonal to this plane is 0.01.

Problem 9.

[10 points]

For a certain 2×2 matrix A, the set

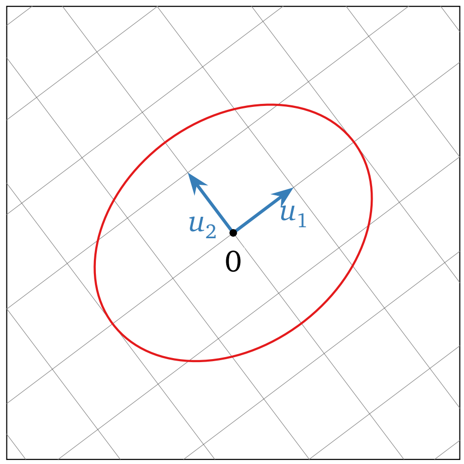
$$\{Ax : \|x\| = 1\}$$

is drawn below.

a) What are the singular values σ_1, σ_2 of A?

$$\sigma_1 = \boxed{2} \qquad \sigma_2 = \boxed{3/2}$$

b) Draw and label the right singular vectors u_1, u_2 of A.



grid lines are 1 unit apart

Problem 10. [10 points]

A certain 2×2 matrix A has the singular value decomposition

$$A = \begin{pmatrix} | & | \\ u_1 & u_2 \\ | & | \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^T,$$

where u_1, u_2, v_1, v_2 are drawn in the diagrams below. Given x and y in the diagram on the left, draw Ax and Ay on the diagram on the right.