MATH 218D-1
PRACTICE MIDTERM EXAMINATION 1

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages**. You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [20 points]

Consider

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ -2 & -1 & -1 & 2 \\ 2 & 3 & 5 & 4 \\ 6 & 3 & -3 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} -2 \\ -8 \\ 4 \\ 0 \end{pmatrix}.$$

a) Carry out Gaussian reduction with maximal partial pivoting to find a $PA = LU$ decomposition. You should obtain

$$U = \begin{pmatrix} 6 & 3 & -3 & 0 \\ 0 & 2 & 6 & 4 \\ 0 & 0 & -4 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Please write the row operations you performed.

b) Write the elementary matrices for the row operations you performed.

c) Solve the equations $Ly = Pb$ and $Ux = y$ to find a solution of $Ax = b$.

d) Briefly explain why step **b)** is faster than solving $Ax = b$ using Gaussian elimination on the augmented matrix $\left(A \mid b\right)$, once you have a $PA = LU$ decomposition.

Solution.

a)

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

b)

$$R_1 \longleftrightarrow R_4 : \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad R_2 \mathrel{+}= \frac{1}{3}R_1 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad R_3 \mathrel{-}= \frac{1}{3}R_1 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_2 \longleftrightarrow R_3 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad R_4 \mathrel{-}= \frac{1}{2}R_2 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 & 1 \end{pmatrix} \qquad R_4 \longleftrightarrow R_3 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R_4 \mathrel{-}= \frac{1}{2}R_3 : \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

c)

$$Ly = Pb \implies y = \begin{pmatrix} 0 \\ 4 \\ -4 \\ -6 \end{pmatrix} \qquad Ux = y \implies x = \begin{pmatrix} 1 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$

d) Gaussian elimination takes about $\frac{2}{3} \cdot 4^3 \approx 43$ flops, whereas forward- and back-substitution take about $4^2 = 16$ flops.

Problem 2. [15 points]

a) Compute the inverse of $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}$.

Be sure to write out any row operations you perform.

b) For which value(s) of k is $\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & k \\ 2 & 3 & 9 \end{pmatrix}$ *not* invertible?

Solution.

a)

$$\begin{pmatrix} 1 & -2 & 3 \\ -2 & 6 & -5 \\ 2 & 3 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} -69 & -27 & 8 \\ -8 & -3 & 1 \\ 18 & 7 & -2 \end{pmatrix}$$

b)

$$k = -\frac{36}{7}$$

Problem 3. [20 points]

Consider

$$A = \begin{pmatrix} 1 & 3 & -2 & 0 \\ -2 & -6 & 6 & -2 \\ 2 & 6 & 3 & -7 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ -8 \\ -10 \end{pmatrix}.$$

a) Find the parametric vector form of the solution set of $Ax = b$. Be sure to write out any row operations you perform.

b) Write down two different solutions of $Ax = b$. (Your answer will be two vectors with numbers in them.)

c) Find a set of vectors spanning the solution set of $Ax = 0$ (for the same matrix A above).

d) Let $v = (-1, 1, 1, 1)$. Check that $Av = 0$, and write v as a linear combination of the spanning vectors you obtained in **c)**.

[**Hint:** what values do the free variables have to take?]

Solution.

a)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

b) Choose any values of the free variables. For instance, $(x_2, x_4) = (1, 0)$ and $(0, 1)$ give $(-5, 1, -2, 0)$ and $(0, 0, -1, 1)$, respectively.

c)

$$\text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

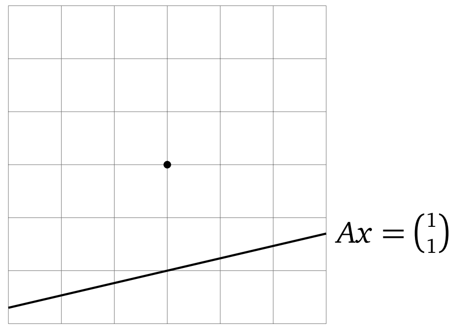
d) One computes $Av = 0$. The second (resp. fourth) coordinate of v is the value of x_2 (resp. x_4), so

$$v = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Problem 4. [20 points]

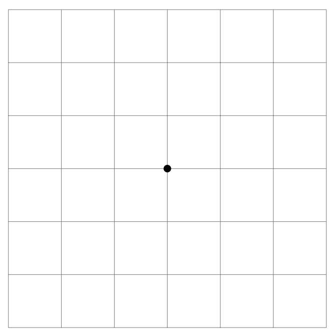
For a certain 2×2 matrix A, the solution set of $Ax = \binom{1}{1}$ is drawn.

a) Draw the solution set of $Ax = 0$ and the solution set of $Ax = \binom{-1}{-1}$ in the grid below. Be sure to label which is which.



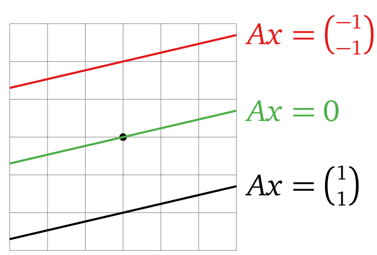
b) $\operatorname{rank}(A) =$ ☐

c) Draw the span of the columns of A. Be precise!



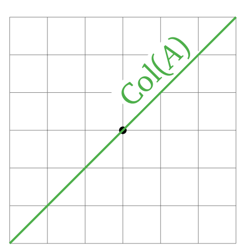
Solution.

a)



c) The rank is 1, since A has one free variable.

d)



Problem 5. [15 points]

Find examples of matrices with the following properties. If no such matrix exists, write "no way, man," or use your favorite colloquialism instead. You need not justify your answers.

a) A 3×5 matrix of rank 4, in RREF.

b) A 2×2 matrix A such that the solution set of $Ax = \binom{3}{4}$ is a line, and $Ax = \binom{1}{1}$ has no solutions.

c) Three vectors in $\mathbf{R}^3$, no two of which are collinear (scalar multiples of each other), that span a plane.

d) A 4×4 matrix A with a pivot in every row such that $A(1, 2, -1, 1) = 0$.

Solution.

a) Yeah, right.

b) There are many answers; one is $A = \begin{pmatrix} 3 & 0 \\ 4 & 0 \end{pmatrix}$.

c) There are many answers; one is $(1, -2, 1), (1, 1, -2), (-2, 1, 1)$.

d) As if.

Problem 6.

[10 points]

Consider the span

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} \right\}.$$

a) Show that $\begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix}$ is in V.

b) Show that $\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$ is not in V.

c) Circle one: V is a point line plane space.

Solution.

a) We solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -4 \end{pmatrix}$$

by row reducing an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 4 & 5 & 6 & -4 \\ 7 & 8 & 9 & -4 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This system has infinitely many solutions, so $(-4, -4, -4)$ is in V.

b) We solve the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$$

by row reducing an augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & -4 \\ 4 & 5 & 6 & -4 \\ 7 & 8 & 9 & 4 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

This system has no solutions, so $(-4, -4, 4)$ is not in V.

c) Since V contains two noncollinear vectors and is not all of $\mathbf{R}^3$, it must be a plane.